U.G. 4th Semester Examination - 2026

MATHEMATICS

[MAJOR-V]

Course Code : MATH-MJ-T-5

(Algebra-II)

[NEP-2020]

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.

GROUP-A

(Marks: 20)

1. Answer any **ten** questions: $2\times10=20$

   a) If G is a finite group and $a \in G$, show that $a^{o(G)} = e$.

   b) Show that every subgroup of a cyclic group is normal.

   c) Prove that the commutator subgroup G' of a group G is a characteristic subgroup of G.

   d) Is the direct product $\mathbb{Z}\times\mathbb{Z}$ a cyclic group? Justify your answer.

   e) Let G be a group and H be a subgroup of G. If $h \in H$, prove that $hH = H$.

   f) Show that there does not exist an onto homomorphism from the group $(\mathbb{Z}_4,+)$ to the Klein's 4-group V.

g) State the Sylow's third theorem.

h) Let R be a ring with unity 1. If $R \neq \{0\}$, show that $0 \neq 1$.

i) Examine if the set $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is a field.

j) Examine if the set $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$ is a subspace of $\mathbb{R}^3$.

k) Determine whether the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x_1, x_2, x_3) = (x_1 + 1, x_2 + 1, x_3 + 1)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$, is a linear transformation or not. Justify your answer.

l) Let P_4 be a real vector space with a basis $\{1, x, x^2, x^3\}$ and $D : P_4 \to P_4$ be defined by $Dp(x) = \frac{d}{dx} p(x)$, $p(x) \in P_4$. Find the matrix of D in the given basis.

m) Prove that $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ is an eigen vector of the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

n) Is the matrix $\begin{pmatrix} 2 & 8 \\ 0 & 2 \end{pmatrix}$ diagonalisable? Justify your answer.

o) Use Cayley-Hamilton theorem to find A^{50}, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

GROUP-B

(Marks: 20)

2. Answer any **four** questions: 5×4=20

a) Find all homomorphisms from the group $\mathbb{Z}_8$ onto the group $\mathbb{Z}_4$.

b) Let H, K be subgroups of a group G and K be normal in G. Prove that $H / H \cap K \cong HK / K$.

c) If G be a non-commutative group with center Z, then prove that the quotient group G/Z is non cyclic.

d) Prove that a finite integral domain is a field.

e) Find the dimension of the subspace $S \cap T$ of the vector space $\mathbb{R}^4$, where $S = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0\}$, $T = \{x, y, z, w) \in \mathbb{R}^4 : 2x + y - z + w = 0\}$.

f) Find a basis for the row space of the matrix

$$\begin{pmatrix} 1 & 0 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 0 & 3 & 1 & 4 \end{pmatrix}.$$

g) Use Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product, spanned by the vectors $(1, 1, 0, 1)$, $(1, -2, 0, 0)$, $(1, 0, -1, 2)$.

GROUP-C

(Marks:20)

3. Answer any **two** questions: $10\times2=20$

a) i) Prove that set $Aut(G)$ of all automorphisms of a group G forms a group. 5

ii) Extend the set $\{1, 1, 0, 0), (1, 1, 1, 0)\}$ to a basis of $\mathbb{R}^4$. 5

b) i) Prove that the order of every subgroup of a finite group G is a divisor of the order of G. 5

ii) Let R be a commutative ring with unity. Prove that an ideal P is a prime ideal if and only if the quotient ring R/P is an integral domain. 5

c) i) Prove that no group of order 30 is simple. 4

ii) Determine the linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3$ which maps the basis vectors $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ of $\mathbb{R}^3$ to $(1, 1, 1)$, $(1, 1, 1)$, $(1, 1, 1,)$ respectively. Verify that $\dim Ker\ T + \dim Im\ T = 3$. 3+3

d) i) Diagonalise the matrix $\begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix}$. 5

ii) Find the number of elements of order 5 in the group $\mathbb{Z}_{25}\times\mathbb{Z}_5$. 5
