U.G. 1st Semester Examination - 2025

MATHEMATICS

[MAJOR]

Course Code : MATH-M-T-1

(Calculus & Analytic Geometry)

[NEP-2020]

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) If $y=(x^2-1)^n$, then prove that

$$(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0.$$

b) Show that $\frac{x}{1+x}<\log(1+x)<x \quad \forall x>0.$

c) For the equiangular spiral $r=ae^{\theta\cot\alpha}$, prove that the radius of curvature subtends a right angle at the pole.

d) Find the curvature of the curve

$$\left(x^2+y^2\right)^2=a^2\left(y^2-x^2\right) \text{ at } (0,a).$$

e) Find the asymptotes of $xy^2-y^2-x^3=0$.

f) Find the equation of the parabola whose focus is the origin and whose directrix is the straight line $2x+y=1$.

g) Does the equation $x^2+3xy+2y^2=0$ represents a pair of straight line? If so what is the angle between them?

[Turn over]

h) Find the nature of the conic

$$\frac{5}{r}=2(1-\cos\theta).$$

Also find its semi-latus-rectum.

i) Find the equation of the sphere which has the points (3, 4, –1) and (–1, 2, 3) as the ends of diameter and also find its radius.

j) Find the equation of the hyperbola whose asymptotes are $2x-y=3$ and $3x+y=7$ and which passes through the point (1, 1).

k) Find the values of c for which the plane $x+y+z=c$ touches the sphere

$$x^2+y^2+z^2-2x-2y-2z-6=0.$$

l) To what points the origin is to be moved so that one can get rid of the first degree terms from the equation $x^2+xy+2y^2-7x-5y+12=0$?

m) Prove that $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{\frac{1}{n}}=\frac{1}{e}$.

n) Prove that $\int_0^{\pi/2}\frac{dx}{2+\cos x}=\frac{\pi}{3\sqrt{3}}$.

o) Trace the curve $ay^2=x^2(a-x)$.

2. Answer any **four** questions: 5×4=20

a) If the pair of straight lines $x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisects the angle between the other pair, then prove that $pq=-1$.

b) Find the shortest distance between the lines $x+4y-2z-5=0$; $x+2y+2z-13=0$ and $2x-2y+z-1=0$; $2x+4y-z-9=0$.

c) A sphere of constant radius d passes through the origin and intersect the coordinate axes in P, Q, R. Prove that the centroid of the triangle PQR lies on the sphere $9(x^2+y^2+z^2)=4d^2$.

d) If $y=\frac{1}{\sqrt{a^2-b^2-c^2}}\cos^{-1}\left(\frac{a\theta-a^2+b^2+c^2}{\theta\sqrt{b^2+c^2}}\right)$ and $\theta=a+b\cos x+c\sin x$, prove that $\frac{dy}{dx}=\frac{1}{\theta}$.

e) Prove that, if $a>0$, then $\lim_{x\to 0+}\frac{x}{a}\left[\frac{b}{x}\right]=\frac{b}{a}$ and $\lim_{x\to 0+}\left[\frac{x}{a}\right]\frac{b}{x}=0$, where $[x]$ is the greatest integer in x but not greater than x. Discuss the left hand limits of these functions.

f) If ρ and ρ' be the radii of curvature at the ends of two conjugate diameters of an ellipse, prove that $\left(\rho^{2/3}+\rho'^{2/3}\right)(ab)^{2/3}=a^2+b^2$.

3. Answer any **two** questions: 10×2=20

a) i) Find $\int\sin^4 x\cos^2 x\,dx$.

ii) If $I_n=\int_0^{\pi/2}x^n\sin x\,dx$ and n > 1, show that

$$I_n+n(n-1)I_{n-2}=n\left(\frac{n}{2}\right)^{n-1}.$$ 5+5

b) i) Prove that

$$\iint\left\{2a^2-2a(x+y)-(x^2+y^2)\right\}dxdy=8\pi a^4,$$

where the region of integration being the region bounded by the circle $x^2+y^2+2a(x+y)=2a^2$.

ii) Differentiate $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$. 5+5

c) i) Show that the pole of any tangent to the hyperbola $xy=c^2$ with respect to the circle $x^2+y^2=a^2$ lies on a concentric and similar hyperbola.

ii) Reduce the following equation in canonical form and determine the nature of the conic:

$$x^2-2xy+2y^2-4x-6y+3=0.$$ 5+5

d) i) Prove that the locus of the foot of the perpendicular from a focus of the conic $\frac{l}{r}=1-e\cos\theta$ on a tangent to it is given by $r^2(1-e^2)-2ler\cos\theta-l^2=0$.

ii) Find the equation of the tangent planes to the ellipsoid $\frac{x^2}{144}+\frac{y^2}{36}+\frac{z^2}{9}=1$ which are parallel to the plane $x-2y-2z+1=0$ and determine the distance between these tangent planes. 5+5
