U.G. 2nd Semester Examination - 2025

MATHEMATICS

[MAJOR]

Course Code : MATH-MJ-T-2

(Algebra-I)

[NEP-2020]

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.

1. Answer any **ten** questions : 2×10=20

a) By Descartes' rule of signs, find the *possible* number of positive real roots of $x^5 - 2x^3 + x - 1 = 0$.

b) Find the order of [8] in the group $(\mathbb{Z}_{12}, +)$.

c) For $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & \lambda \end{pmatrix}$, find all λ such that A is invertible.

d) Find the last digit of 3^{123}.

e) Find $\varphi(4800)$.

f) Find all complex numbers z such that $|z| = \bar{z} + 2i$.

[Turn over]

g) Find the general and the principal value of 3^i.

h) Find the modulus and the principal amplitude of $z = 1+\cos 2\theta - i\sin 2\theta$, where $\frac{\pi}{2} < \theta < \pi$.

i) Obtain the condition on the constants p and q such that $x^3 - 3px + q = 0$ has a repeated root.

j) Prove that the equation $1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}=0$ cannot have a multiple root.

k) In S_4, let $\sigma = (1\ 4\ 2\ 3)$. Find $(1\ 2)\sigma(1\ 2)^{-1}$.

l) On $\mathbb{Z}\times\mathbb{Z}$, define $(a,b)\sim(c,d)$ iff $a-d=b-c$. Examine whether $\sim$ is an equivalence relation or not?

m) Let $[A\,|\,\mathbf{b}] = \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & t \end{array}\right)$. Find t for which the system is consistent.

n) Write differences between *echelon form* and *normal form* (under row / column equivalence).

o) Wthat is the difference between reduced row echelon form and row echelon form? Give examples.

2. Answer any **four** questions: 5×4=20

a) Show that $\operatorname{Log}\left((1-i)^{\frac{1}{2}}\right) = \frac{1}{2}\operatorname{Log}(1-i)$.

b) If α, β, γ are the roots of $x^3-5x+2=0$, determine the value of $((\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$.

c) Using row and column operations, reduce $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix}$ to its normal form and find rank(A).

d) Find the equation whose roots are the squares of the roots of the equation $x^3+x+10=0$. Then, apply Descartes' rule of signs determine the exact number of real and complex roots of the given equation.

e) Determine all elements of order 2 in U_{20} (mod 20).

f) In S_3, let $H = \langle(12)\rangle$ and $K = \langle(123)\rangle$. Compute HK and $H\cap K$. Verify $|HK| = \frac{|H||K|}{|H\cap K|}$ and decide whether $HK = S_3$.

3. Answer any **two** questions: 10×2=20

a) i) Solve the equation $x^4 - x^3 - 4x^2 - 5x - 3 = 0$, given that two of its roots, say α and β satisfy the relation $3\alpha + \beta = 0$.

ii) Find the greatest value of $|z|$ satisfying the relation $|z - 4/z| = 2$. 5+5

b) i) If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find the equation whose roots are

$$\alpha\beta - \gamma^2, \quad \beta\gamma - \alpha^2, \quad \gamma\alpha - \beta^2.$$

ii) Use the Euclidean algorithm find gcd(987, 610) and find integers x, y such that $987x + 610y = \gcd(987, 610)$. 5+5

c) i) State and prove the Lagrange's theorem.

ii) Classify all subgroups of $\mathbb{Z}_{24}$ and list their generators.

iii) Determine the number of generators of $\mathbb{Z}_{24}$. 4+4+2

d) i) Reduce $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ to row-reduced echelon form and solve $Ax = b$ if possible for $\mathbf{b} = (1, 2, 1)^T$. Find rank(A).

ii) Prove that,

$$\operatorname{Sin}^{-1}(x) = (-1)^n \left(\frac{\pi}{2} - i \ln\left(x + \sqrt{x^2 - 1}\right)\right) + \pi n,$$

$n \in \mathbb{Z}$, where $x > 1$ be a real number.

5+5
