U.G. 4th Semester Examination - 2020

MATHEMATICS

[HONOURS]

Course Code : MTMH-CC-T-9

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Define differentiability of a function of two variables at a point.

b) Examine whether $f(x,y)=|x|(1+y)$ is differentiable at (0, 0).

c) If $f(x,y)=\log\left(x^3+y^3-x^2y-xy^2\right)$, then show that $f_x+f_y=\frac{2}{x+y}$.

d) Find the direction in which the function $f(x,y)=x^2y+e^{xy}\sin y$ increases most rapidly at (1, 0).

e) Find critical points of the function $f(x,y)=3x^2(y-1)+y^2(y-3)+1$.

[Turn Over]

f) State Stoke's theorem.

g) Show that the vector $\vec{F}=xy\hat{i}+zx\hat{j}-yz\hat{k}$ is solenoidal.

h) Evaluate $\int_0^1\int_0^1\int_0^1(x+y+z)\,dxdydz$.

i) Find gradient of the function $\phi=x^2y+2z$ at the point (2, 1, 0).

j) If $u=\sqrt{x^2+y^2+z^2}$, show that $u_{xx}+u_{yy}+u_{zz}=\frac{2}{u}$.

k) State Young's theorem for the equality of mixed partial derivatives.

l) If $\vec{\nabla}f(x,y,z)=2xyze^{x^2}\hat{i}+ze^{x^2}\hat{j}+ye^{x^2}\hat{k}$ and $f(0,0,0)=7$, evaluate $f(1,1,2)$.

m) If f(x, y) = k and S = [a, b]×[c, d], where k is a constant, then find the value of $\iint_S f(x,y)\,dxdy$.

n) Prove that $\vec{F}=\left(y^2\cos x+z^3\right)\hat{i}+\left(2y\sin x-4\right)\hat{j}+\left(3xz^2+2\right)\hat{k}$ is a conservative force field.

o) Let C be the boundary of the square $V=[0,1]\times[0,1]$ oriented counter clockwise. Evaluate $\int_C\left\{\left(y^4+x^3\right)dx+2x^6dy\right\}$, using Green's theorem.

p) Write an equivalent integral of

$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dxdy$$ with the order of integration reversed.

2. Answer any **four** questions: 5×4=20

a) Let (a, b) be an interior point of domain of definition of a function f of two variables x, y. If $f_x(a, b)$ exists and $f_y(x, y)$ is continuous at (a, b), then prove that f(x, y) is differentiable at (a, b).

b) If $f(x,y)=\begin{cases} xy, & |x|\ge|y| \\ -xy, & |x|<|y| \end{cases}$. Show that $f_{xy}(0,0)\neq f_{yx}(0,0)$.

c) Find the total differentials of the first and second order of the function

$z=2x^2-3xy-y^2$.

d) Use Lagrange multipliers to evaluate the maximum value of the function

$f(x,y,z)=x^2+y^2+z^2$ subject to the constraint $x^4+y^4+z^4=1$.

e) Find the directional derivative of $f(x,y,z)=x^2+2y^2-3z^2$ at (1, 1, 1) along the line joining the points (0, 0, 0) and (1, 1, 1).

f) Let F = (P,Q) be a continuously differentiable function defined on a simply connected region D in R^2. Show that $\int_C Pdx+Qdy=0$ around every piecewise smooth closed curve C in D if and only if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, for all $(x,y)\in D$.

3. Answer any **two** questions: 10×2=20

a) i) If (a, b) be a point in the domain of definition of f(x, y) such that $f_x(x, y)$ and $f_y(x, y)$ are differentiable at (a, b), prove that $f_{xy}(a, b) = f_{yx}(a, b)$. 5

ii) If a function f(x, y) of two variables x and y when expressed in terms of new variables u and v defined by $x=\frac{1}{2}(u+v)$ and $y^2=uv$ becomes g(u, v), then show that

$$\frac{\partial^2 g}{\partial u\,\partial v}=\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}+\frac{2x}{y}\frac{\partial^2 f}{\partial x\,\partial y}+\frac{\partial^2 f}{\partial y^2}+\frac{1}{y}\frac{\partial f}{\partial y}\right).$$

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b) i) Let f : [a, b] → R and g : [c, d] → R be continuous and h : U→ R be defined as h(x, y) = max {f(x), g(y)} for all $(x,y)\in U$, where $U=\{(x,y): a \le x \le b, c \le y \le d\}$.

Prove that h(x, y) is continuous in U. 5

ii) Let $\frac{4}{x}+\frac{9}{y}+\frac{16}{z}=25$. Use Lagrange's method to find the values of x, y, z such that x+y+z is minimum. 5

c) i) Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=L$ using the spherical polar coordinates. 5

ii) Evaluate $\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(\frac{2x-y}{2}\right) dxdy$ by applying the transformation $x=u+v, y=2v$. 5

d) i) Show that $\vec{F}=(2xy+z^3)\hat{i}+x^2\hat{j}+3xz^2\hat{k}$ is a conservative field. Find the scalar potential ϕ(x, y, z) such that $\vec{F}=\vec{\nabla}\phi$. 5

ii) Verify Green's theorem in a plane for $\int_M \{(x^2+xy)dx+xdy\}$, where M is the curve enclosing the region bounded by $y=x^2$ and $y=x$. 5

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