

U.G. 3rd Semester Examination - 2019

MATHEMATICS

[HONOURS]

Course Code : MATH(H)CC-05-T

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**Candidates are required to give their answers in their own words as far as practicable.**Symbols and notations have their usual meanings.*1. Answer any ten questions: $2 \times 10 = 20$

- i) Use sequential criterion for limits to show that the following limit does not exist

$$\lim_{x \rightarrow 0} \frac{1}{x} \sin \frac{1}{x}.$$

- ii) Give example of function f and g which are not continuous at a point $c \in \mathbb{R}$ but the sum $f+g$ is continuous at c .

- iii) Using ϵ - δ definition, show that

$$\lim_{x \rightarrow \infty} \frac{[x]}{x} = 1.$$

[Turn over]

- iv) Verify whether $(\mathbb{R}, d)$ is a metric space, where $d(x, y) = |x^2 - y^2|$, $\forall x, y \in \mathbb{R}$.
- v) Define diameter of a set in a metric space (X, d) .
- vi) Does $f'(c) = 0$ always imply existence of an extremum of f at c ? Justify.
- vii) Give an example of a function which has a jump discontinuity in its domain of definition.
- viii) Show that the equation $f(x) = xe^x - 2$ has a root in $[0, 1]$.
- ix) Expand $\log \sin(x+h)$ in power of h by Taylor's Theorem.
- x) Give geometrical interpretation of Lagrange's Mean Value Theorem.
- xi) Define limit point of a set in a metric space (X, d) . Give one example.
- xii) For a metric space X , show that a point $a \in X$ is a cluster point of $A \subset X$ if there exists $\{a_n\}_{n=1}^{\infty}$ in A such that $\lim_{n \rightarrow \infty} a_n = a$.

xiii) Show that there does not exist a function ϕ such that $\phi'(x) = f(x)$, where $f(x) = x - [x]$, $x \in [0, 2]$.

xiv) Discuss the applicability of Rolle's theorem for $f(x) = 2 + (x-1)^{\frac{2}{3}}$ in $[0, 2]$.

xv) Show that $f(x) = |x+2|$ is continuous at $x = -2$ but not differentiable at this point.

2. Answer any **four** questions: $5 \times 4 = 20$

i) Let f be a continuous function on $[a, b]$ and c be any real number between $f(a)$ and $f(b)$, then show that there exist a real number x in (a, b) such that $f(x) = c$.

Construct an example to show that continuity of f is not necessary for the existence of such x as above. $3+2$

ii) State and prove Rolle's theorem.

iii) Find the maxima and minima of the function

$$f(x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \text{ for all } x \in [0, \pi].$$

iv) Define a Metric space show that $(\mathbb{R}^2, d)$ is a metric space, where the metric $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$; $x, y \in \mathbb{R}^2$ when $x = (x_1, x_2), y = (y_1, y_2)$.

v) Suppose n -th derivative of a function f exists finitely in a closed interval $[a, a+h]$. Then show that there exists a positive proper fraction θ satisfying the relation

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h).$$

vi) Let $f: [a, b] \rightarrow \mathbb{R}$ be such that f has a local extremum as an interior point c of $[a, b]$. If $f'(c)$ exists, then prove that $f'(c) = 0$.

Answer any **two** questions from **Question No. 3** to **Question No. 6:** 10×2=20

3. a) A function $f : [0, 1] \rightarrow [0, 1]$ is continuous on $[0, 1]$.
Prove that there exists a point c in $[0, 1]$ such that $f(c)=c$.

b) Show that every uniformly continuous function on an interval is continuous on that interval, but the converse is not true.

c) Prove that the function $f(x) = \frac{1}{x}$, $x \in (0, 1]$ is not uniformly continuous on $(0, 1]$.

3+(2+2)+3

4. a) If f' and g' exist for all $x \in [a, b]$ and $g'(x) \neq 0 \forall x \in (a, b)$, then prove that for some

$$c \in (a, b), \frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$

b) Obtain the Maclaurin's series expansion of $\log(1+x)$, $-1 < x \leq 1$.

c) Show that

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, \quad x > 0.$$

3+4+3

5. a) Prove that in a metric space every open ball is an open set and every closed ball is a closed set.

b) Define the following with example:

i) Subspace of a metric space

ii) Separable metric space (3+3)+(2+2)

6. a) Show that the function f on $[0, 1]$ defined as

$$f(x) = \frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}, \quad n = 0, 1, 2, \dots,$$

$f(0)=0$ is discontinuous at $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \dots$

b) Show that $\lim_{x \rightarrow \infty} a^x \cdot \sin \frac{b}{a^x} = \begin{cases} 0 & \text{if } 0 < a < 1 \\ b & \text{if } a > 1 \end{cases}$

- c) Prove that in a metric space (X, d) , the interior of a set $A \subset X$ is the largest open subset of A .

3+3+4
