

SOUND

A PHYSICAL TEXT BOOK FOR PASS AND
HONOURS DEGREE CLASSES

By

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Second Revised Edition

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IN MEMORY OF MY PARENTS

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PREFACE TO THE SECOND REVISED EDITION

In this edition some portions of a few chapters have been rewritten, substantial alterations made here and there and several new topics introduced and discussed. In effecting the changes, suggestions received from learned teachers of Physics were kept in mind. It is hoped that the book in the present form will prove more useful to the students.

The author expresses his deep sense of gratitude towards his well-wishers for the encouragement and support received from them.

Serampore College.

K. Bhattacharyya.

6. 6. 67.

PREFACE TO THE REVISED EDITION

In this edition the chapter on coupled vibration has been practically rewritten with a view to making more clear the fundamental principles involved. In the appendix, a new topic on acoustic impedance has been discussed. A few additions and alterations have been made here and there with the hope that they will prove to be of advantage.

The author is grateful to Prof. S. N. Sarker, M. A. for the help he rendered by going through some proofs and to Prof. G. D. Bhattacharyya, M. sc. for kindly taking the trouble of checking a few mathematical calculations. He is specially indebted to his colleagues of the department of Physics, Serampore College, for the encouragement and support received from them. He also wishes to express his deep sense of gratitude to those learned professors who made favourable comments on the book, offered constructive suggestions and recommended it to the students.

Finally, Sri Joydeb Ganguly, B. A. (Hons.), Proprietor, Hindustan Publishing Concern must be thanked for expediting publication of the revised edition.

Serampore College.

K. Bhattacharyya.

8. 11. 65.

PREFACE TO THE FIRST EDITION

The book is specially prepared keeping in mind the needs of the students preparing for an Honours Degree in Physics of Indian Universities. Attempts have been made to treat the subject analytically and to clarify those "easy" things usually taken for granted by the students, but seldom clearly understood. The author feels no hesitation to acknowledge that in writing of the book most of the available standard text books and treatises on sound have been freely consulted and made use of.

The author acknowledges with gratitude the kind help and encouragement received from his colleagues without which the conception could not have been possibly realised. He is indebted in particular to Shri G. D. Bhattacharyya, M. Sc., Shri Sukhendra Dey, M. Sc. and Shri T. D. Mazumdar, M. A., all esteemed colleagues in the Department of Mathematics, Serampore College, for going through most of the proofs and checking mathematical calculations contained there in. He is specially beholden to Shri Parimal Kanti Ghose, M. Sc., Department of Applied Mathematics, University of Calcutta, for looking through a portion of the manuscript and offering a number of welcome and valuable suggestions. Thanks are also undoubtedly due to Shri Joydeb Ganguly, Proprietor, Hindustan Publishing Concern, Calcutta, for all he has done to have the book published in so short a time.

The author regrets several errors which could not be rectified due to conditions beyond control. An errata list, however, has been appended for their detection and rectification.

The author will be grateful for any constructive suggestions towards the improvement of the book.

Serampore College.
16.9.63.

K. Bhattacharyya.

CHAPTER I

SIMPLE HARMONIC MOTION

1. **General equations :** In the phenomena studied under "Sound" we have to deal with periodic motions. The simplest of all periodic motions is *simple harmonic motion*. In this type of motion a system vibrates about the mean position of rest and the displacement is a circular function of time.

Suppose a particle has mass m and its displacement at any instant is x from the initial position of rest. If the force tending to restore the particle to its initial position of rest is proportional to the displacement and in a direction opposite to it, we can write as equation of motion

$$m \frac{d^2 x}{dt^2} = -sx \text{ where } s \text{ is a constant called "stiffness" constant}$$

which is the force required to produce unit displacement of the particle from its initial position of rest.

$$\text{Writing } \frac{s}{m} = n^2$$

$$\frac{d^2 x}{dt^2} + n^2 x = 0 \quad \dots \quad (1)$$

The above is an equation of second order. Let $x = Ae^{\alpha t}$ be a solution of the equation.

$$\text{Then } \frac{dx}{dt} = \alpha Ae^{\alpha t}, \quad \frac{d^2 x}{dt^2} = \alpha^2 Ae^{\alpha t}$$

$$\text{Substituting the value of } \frac{d^2 x}{dt^2} \text{ in (1)}$$

$$\text{we have } Ae^{\alpha t} (\alpha^2 + n^2) = 0.$$

Since $e^{\alpha t}$ cannot be zero for all values of t and A is not zero

$$\alpha^2 + n^2 = 0$$

$$\text{or } \alpha = \pm in$$

Hence solution may be $x = A_1 e^{int}$ or, $x = A_2 e^{-int}$.

Also equation (1) will be satisfied when

$$x = Ae^{int} + Be^{-int} \quad \dots \quad (2)$$

where A and B are two constants to be determined from initial conditions.

Expression (2) can be written as $x = A(\cos nt + i \sin nt) + B(\cos nt - i \sin nt) = (A+B) \cos nt + i(A-B) \sin nt$

Now A and B may be real or complex quantities containing real and imaginary parts. But since x is real we can write

$$x = a_1 \cos nt + b_1 \sin nt \quad \dots \quad (3)$$

where a_1 and b_1 are the real parts of the coefficients of $\cos nt$ and $\sin nt$ respectively.

Writing $a_1 = a \cos \epsilon$, $b_1 = a \sin \epsilon$

we have $x = a \cos \epsilon \cos nt + a \sin \epsilon \sin nt$

$$= a \cos (nt - \epsilon) \quad \dots \quad (4)$$

where a and ϵ are constants given by

$$a^2 = a_1^2 \cos^2 \epsilon + a_1^2 \sin^2 \epsilon = a_1^2 + b_1^2$$

$$\text{and } \tan \epsilon = \frac{b_1}{a_1}$$

Hence the maximum possible displacement in the positive direction is $x_m = a$ when $nt - \epsilon$ is $0, 2\pi, 4\pi$ etc.; the magnitude of the maximum in the negative direction is also equal to a when $nt - \epsilon = \pi, 3\pi, 5\pi$ etc. Thus the particle oscillates between two points which are at a distance a apart from the mean position; the quantity a is known as the *amplitude* of vibration. The same displacement repeats after an interval of time T called *time period* given by $nT = 2\pi$.

Or, the time period of oscillation

$$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{\frac{s}{m}}} = 2\pi \sqrt{\frac{m}{s}}$$

Thus $T = 2\pi \sqrt{\frac{\text{mass of the particle}}{\text{Restoring force per unit displacement}}}$

Since T is the time required for one oscillation, the number of oscillations per second $N = \frac{1}{T}$

$$= \frac{1}{2\pi} \sqrt{\frac{s}{m}}$$

The meaning of the expression (4) can be made clear when we consider the motion of a point P moving with a uniform angular velocity n in a circular path of radius a .

Suppose the particle describing anticlockwise motion is at A_1 at time $t=0$; AOB is a fixed diameter and angle $AOA_1 = \epsilon$. Let P be the position of the point at any instant t . Angle $POA_1 = nt$. Hence the projection OM of OP on AOB is $OM = x = OA \cos \angle POM = a \cos (nt - \epsilon)$.

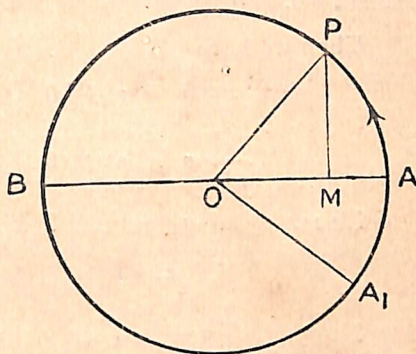


Fig. 1

The particle is behind the fixed diameter by angle ϵ at the starting position. The angle ϵ gives the *phase* of the particle at the instant of start and is known as the *epoch* or the *initial phase*.

Evaluation of constants :

(i) Let us return to (3) i. e., $x = a_1 \cos nt + b_1 \sin nt$ and see how a_1 and b_1 are determined.

Suppose the particle is brought to a distance x_1 and then released. Thus x_1 is the maximum displacement and if we count time from the instant the particle is released, we have $x = x_1$ at time $t=0$

$$\text{Hence, } x_1 = a_1 \cos 0 + b_1 \sin 0 \\ = a_1$$

Again the velocity at the instant of release must also be zero ;

$$\text{now velocity } \frac{dx}{dt} = -na_1 \sin nt + nb_1 \cos nt$$

Hence $0 = nb_1$

Since $n \neq 0$, $b_1 = 0$,

Thus $x = x_1 \cos nt$... (5)

Again let us suppose that an impulse is given to the particle at its position of rest so that the velocity of the particle is v at time $t=0$.

Then we have

$$x=0 \quad \text{at } t=0$$

$$\frac{dx}{dt}=v \quad \text{at } t=0$$

Since, $x = a_1 \cos nt + b_1 \sin nt$

$$\therefore a_1 = 0$$

Now $\frac{dx}{dt} = -na_1 \sin nt + nb_1 \cos nt$

$$\text{At } t=0, \frac{dx}{dt}=v,$$

$$\therefore v = nb_1$$

$$\text{or, } b_1 = \frac{v}{n}$$

Hence, $x = \frac{v}{n} \sin nt$... (6)

Thus we see that a_1 and b_1 in the general solution (3) can be determined from initial conditions.

(ii) Let us consider the general case when x_1 and v_1 are the displacement and velocity of the body at any given instant t_1 .

Let the motion of the body be represented by

$$x = a \cos n\tau + b \sin n\tau \quad \dots (7)$$

counting time $\tau=0$ when time $t=t_1$.

$$\text{Now } \frac{dx}{d\tau} = \frac{dx}{dt} = -na \sin n\tau + nb \cos n\tau \quad \dots (8)$$

$$\text{At } \tau=0 \text{ i. e., at } t=t_1, x=x_1, \frac{dx}{dt}=v_1.$$

Putting $\tau=0$, in expressions (7) and (8)

$$x_1 = a, \quad v_1 = nb \quad \text{or } b = \frac{v_1}{n}$$

Hence, substituting in (7), we have,

$$x = x_1 \cos n\tau + \frac{v_1}{n} \sin n\tau$$

But at an instant t , $\tau = t - t_1$; hence displacement at an instant t is

$$x = x_1 \cos n(t-t_1) + \frac{v_1}{n} \sin n(t-t_1) \quad \dots (9)$$

If in the expression (9), we apply the relevant conditions, we shall obtain expressions (5) and (6).

2. Energy at any instant: Let us suppose that the particle has a displacement x at any instant. The opposing force at this displacement is sx . If the displacement is increased by dx the work done against this force is $sx \cdot dx$ and this is the increase of the potential energy for a displacement dx . Hence the total potential energy for a displacement x is

$$\int_0^x sx \cdot dx = \frac{sx^2}{2} \quad \dots (10)$$

Kinetic energy at this instant is

$$\frac{1}{2}m \left(\frac{dx}{dt} \right)^2$$

In an ideal system where there is no loss of energy due to friction etc., sum of kinetic and potential energies is constant and is equal to the total energy of the system.

From (10), at $x=0$, $P. E. = 0$

and hence kinetic energy is maximum. Again at maximum displacement which is equal to amplitude of vibration, potential energy is maximum whereas kinetic energy is zero.

Deduction of Equation of motion from Principles of energy :

Since total energy is constant

$$\frac{d}{dt} \left\{ \frac{sx^2}{2} + \frac{1}{2}m \left(\frac{dx}{dt} \right)^2 \right\} = 0$$

$$\frac{s}{2} \cdot 2x \cdot \frac{dx}{dt} + \frac{1}{2}m \cdot 2 \cdot \frac{dx}{dt} \frac{d^2x}{dt^2} = 0$$

$$\frac{dx}{dt} \left\{ m \frac{d^2x}{dt^2} + sx \right\} = 0.$$

Since $\frac{dx}{dt}$ cannot be zero for all values of t

$$m \frac{d^2x}{dt^2} + sx = 0$$

and we arrive at the same differential equation of simple harmonic motion,

From $x = a \cos (nt - \epsilon)$

$$\left(\frac{dx}{dt} \right)^2 = n^2 a^2 \sin^2 (nt - \epsilon).$$

$\therefore$ Maximum kinetic energy

$$= \frac{1}{2}m \left(\frac{dx}{dt} \right)_{max}^2 = \frac{1}{2}mn^2 a^2.$$

Maximum potential energy

$$= \frac{s(x_{max})^2}{2} = \frac{s \cdot a^2}{2}.$$

The average kinetic and potential energies may also be calculated in the following way

$$\text{Average kinetic energy} = \frac{\frac{1}{2}m \int_0^T \left(\frac{dx}{dt} \right)^2 dt}{T}$$

$$\begin{aligned} &= \frac{1}{2T} m \int_0^T n^2 a^2 \sin^2 (nt - \epsilon) dt \\ &= \frac{mn^2 a^2}{2T} \int_0^T \frac{1 - \cos 2(nt - \epsilon)}{2} dt \\ &= \frac{mn^2 a^2}{4T} \left[t - \frac{1}{2n} \sin 2(nt - \epsilon) \right]_0^T \\ &= \frac{mn^2 a^2}{4T} [T] - \frac{mn^2 a^2}{8Tn} \left[\sin 2 \left(\frac{2\pi}{T} \cdot T - \epsilon \right) - \sin 2(-\epsilon) \right] \\ &= \frac{mn^2 a^2}{4} - \frac{mn^2 a^2}{8Tn} (-\sin 2\epsilon + \sin 2\epsilon) \\ &= \frac{mn^2 a^2}{4} \dots \quad (11) \end{aligned}$$

Thus the average kinetic energy of the particle is half the maximum kinetic energy.

Potential energy, when displacement is x , is $\frac{sx^2}{2}$

Hence average potential energy over an oscillation is

$$\begin{aligned} &\frac{1}{T} \int_0^T \frac{sx^2}{2} dt \\ &= \frac{s}{2T} \int_0^T a^2 \cos^2 (nt - \epsilon) dt \\ &= \frac{sa^2}{2T} \int_0^T \left[\frac{1 + \cos 2(nt - \epsilon)}{2} \right] dt \\ &= \frac{sa^2}{2T} \cdot T/2 = \frac{sa^2}{4} \dots \quad (12) \end{aligned}$$

which is equal to half the maximum potential energy.

3. Superposition of simple harmonic motions :

(a) Motions along same straight line of same periods but different phases and amplitudes :

Suppose the two vibrations are given by $x_1 = a_1 \cos (nt - \epsilon_1)$ and $x_2 = a_2 \cos (nt - \epsilon_2)$.

Hence the resultant vibration is

$$\begin{aligned} x &= x_1 + x_2 = a_1 \cos (nt - \epsilon_1) + a_2 \cos (nt - \epsilon_2) \\ &= a_1 \cos nt \cos \epsilon_1 + a_1 \sin nt \sin \epsilon_1 \\ &\quad + a_2 \cos nt \cos \epsilon_2 + a_2 \sin nt \sin \epsilon_2 \\ &= (a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2) \cos nt + (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2) \sin nt \end{aligned}$$

Let $a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 = A \cos \delta$

and $a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2 = A \sin \delta$

where A and δ are new constants such that

$$\begin{aligned} A^2 &= (a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2)^2 + (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2)^2 \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos (\epsilon_1 - \epsilon_2) \end{aligned}$$

$$\text{and } \tan \delta = \frac{a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2}{a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2}$$

$$\text{Hence } x = A \cos (nt - \delta) \quad \dots \quad (13)$$

Thus the resultant vibration has the same period as that of component vibrations.

If instead of two vibrations there are several vibrations of different amplitudes and phases but of same time period, the resultant vibration can be likewise deduced.

Hence $x = a_1 \cos (nt - \epsilon_1) + a_2 \cos (nt - \epsilon_2) + a_3 \cos (nt - \epsilon_3) + \dots$

$$= (a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 + \dots) \cos nt$$

$$+ (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2 + \dots) \sin nt$$

$$\text{Putting } a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 + a_3 \cos \epsilon_3 + \dots = A \cos \delta$$

$$\text{and } a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2 + a_3 \sin \epsilon_3 + \dots = A \sin \delta$$

$$\text{we have } x = A \cos (nt - \delta)$$

$$\begin{aligned} \text{where } A^2 &= (a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 + a_3 \cos \epsilon_3 + \dots)^2 \\ &\quad + (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2 + a_3 \sin \epsilon_3 + \dots)^2 \end{aligned}$$

$$\text{and } \tan \delta = \frac{a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2 + \dots}{a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 + \dots}$$

(b) Two vibrations of slightly different frequencies along same straight line : Beats :

$$\text{Let } x_1 = a_1 \cos (nt - \epsilon_1)$$

$$\text{and } x_2 = a_2 \cos \{(n+m)t - \epsilon_2\}$$

$$\text{We can write } N_1 = \frac{n}{2\pi} \text{ and } N_2 = \frac{n+m}{2\pi}$$

where N_1 and N_2 are the frequencies of the two vibrations.

Putting $-mt + \epsilon_2 = \epsilon'_2$ for the phase term of the second vibration we have

$$x_1 = a_1 \cos (nt - \epsilon_1)$$

$$\text{and } x_2 = a_2 \cos (nt - \epsilon'_2)$$

Compounding the two vibrations as in art. 3(a)

$$x = x_1 + x_2 = A \cos (nt - \delta)$$

$$\text{where } \tan \delta = \frac{a_1 \sin \epsilon_1 + a_2 \sin \epsilon'_2}{a_1 \cos \epsilon_1 + a_2 \cos \epsilon'_2}$$

$$\begin{aligned} \text{and } A^2 &= a_1^2 + a_2^2 + 2a_1 a_2 \cos (\epsilon_1 - \epsilon'_2) \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos (\epsilon_1 + mt - \epsilon_2) \end{aligned}$$

Thus when $\epsilon_1 - \epsilon_2 + mt = (2s+1)\pi$ where $s=0, 1, 2, 3$ etc.,

$$A = a_1 - a_2$$

$$\text{and when } \epsilon_1 - \epsilon_2 + mt = 2s\pi$$

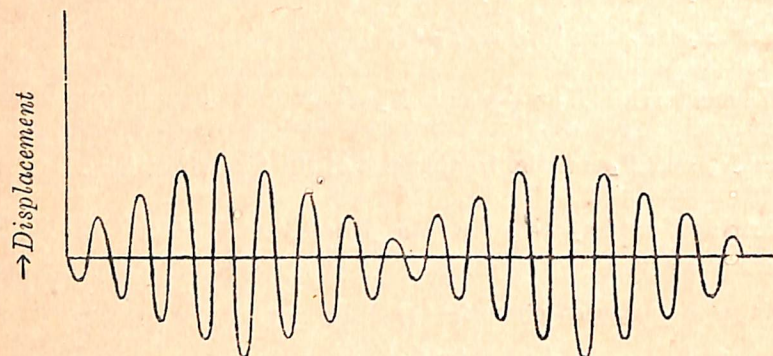
$$A = a_1 + a_2$$

Hence the amplitude of the resultant vibration changes between the limits $a_1 - a_2$ to $a_1 + a_2$.

If $a_1 = a_2$, then the limits would have been 0 and $2a$. Thus the amplitude of the resultant vibration changes periodically with a frequency equal to $\frac{m}{2\pi}$ which is the difference of the two component frequencies.

This phenomenon is known as the *beats* and is observed when two tuning forks or any two sources of sound of nearly equal

frequencies are sounded together. The method of beats is a very important one in the measurement of an unknown frequency.



→ Time

Fig. 2

The displacement curve of a particle on which two vibrations of nearly equal frequencies act along the same straight line is given above.

(c) **Superposition of two vibrations in a plane, at right angles to each other, time periods being equal :**

Let the vibrations be

$$x = a_1 \cos (nt - \epsilon_1)$$

$$\text{and } y = a_2 \cos (nt - \epsilon_2)$$

Displacement along y at any instant can be written as,

$$y = a_2 \cos (nt - \epsilon_1 + \epsilon_1 - \epsilon_2)$$

$$= a_2 \cos (nt - \epsilon_1) \cos (\epsilon_1 - \epsilon_2)$$

$$- a_2 \sin (nt - \epsilon_1) \sin (\epsilon_1 - \epsilon_2)$$

$$= a_2 \cdot \frac{x}{a_1} \cdot \cos (\epsilon_1 - \epsilon_2)$$

$$- a_2 \times \sqrt{1 - \frac{x^2}{a_1^2}} \times \sin (\epsilon_1 - \epsilon_2)$$

$$= \frac{a_2}{a_1} x \cos (\epsilon_1 - \epsilon_2) - \frac{a_2}{a_1} \sqrt{a_1^2 - x^2} \sin (\epsilon_1 - \epsilon_2)$$

$$\text{Or, } \sqrt{a_1^2 - x^2} \sin (\epsilon_1 - \epsilon_2) = x \cos (\epsilon_1 - \epsilon_2) - y \cdot \frac{a_1}{a_2}$$

Squaring,

$$(a_1^2 - x^2) \sin^2 (\epsilon_1 - \epsilon_2) = x^2 \cos^2 (\epsilon_1 - \epsilon_2) + y^2 \cdot \frac{a_1^2}{a_2^2} - \frac{2a_1}{a_2} xy \cos (\epsilon_1 - \epsilon_2)$$

Re-arranging,

$$a_1^2 \sin^2 (\epsilon_1 - \epsilon_2) = x^2 + y^2 \cdot \frac{a_1^2}{a_2^2} - \frac{2a_1}{a_2} xy \cos (\epsilon_1 - \epsilon_2)$$

$$\text{Or, } \sin^2 (\epsilon_1 - \epsilon_2) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2yx}{a_1 a_2} \cos (\epsilon_1 - \epsilon_2) \dots (14)$$

Thus the motion in general is elliptical and the position of the particle at any instant depends on ϵ_1 , ϵ_2 , a_1 and a_2 .

Case (1) Let $\epsilon_1 - \epsilon_2 = 0$,

Then (14) reduces to

$$0 = \frac{x}{a_1} - \frac{y}{a_2}$$

This represents a straight line passing through the origin and making an angle θ with the +ve direction of X-axis such that

$$\tan \theta = \frac{a_2}{a_1}$$

Case (2) If $\epsilon_1 - \epsilon_2 = \pi$,

$$\frac{x}{a_1} + \frac{y}{a_2} = 0$$

This also represents a straight line passing through the origin.

Case (3) If $\epsilon_1 - \epsilon_2 = \frac{\pi}{2}$

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$$

This represents an ellipse with two axes coinciding with the X and Y axes.

Case (4) $a_1 = a_2 = a$ and $\epsilon_1 - \epsilon_2 = \frac{\pi}{2}$

then $x^2 + y^2 = a^2$ which represents a circular path with radius a .

(d) Two vibrations of slightly different frequencies at right angles to each other :

Let the two vibrations be

$$x = a_1 \cos (nt - \epsilon_1)$$

$$y = a_2 \cos \{(n+m)t - \epsilon_2\}$$

Writing $-mt + \epsilon_2 = \delta$

$$y = a_2 \cos (nt - \delta)$$

As deduced in the previous article we have

$$\sin^2 (\epsilon_1 - \delta) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2yx}{a_1 a_2} \cos (\epsilon_1 - \delta)$$

Shape of the resultant path will depend on the value of $\epsilon_1 - \delta$. If $\epsilon_1 - \delta = s\pi$ where s is an integer, the paths will be straight lines; if it is $(2s + \frac{\pi}{2})$ the resultant curve will be an ellipse with axes along the directions of vibrations. In the general case the form of the curve will be elliptical. With time, $\delta = \epsilon_2 - mt$ will change and hence trace of the path described by the particle will gradually change its pattern and the greater the difference between the two time periods, the more quickly the nature of the curve will change.

At any instant t

$$\sin^2 (\epsilon_1 - \epsilon_2 + mt) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2yx}{a_1 a_2} \cos (\epsilon_1 - \epsilon_2 + mt)$$

At any other later instant t_1

$$\sin^2 (\epsilon_1 - \epsilon_2 + mt_1) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2yx}{a_1 a_2} \cos (\epsilon_1 - \epsilon_2 + mt_1)$$

The two curves will be identical,

$$\text{if } \epsilon_1 - \epsilon_2 + mt_1 = 2\pi + \epsilon_1 - \epsilon_2 + mt$$

$$\text{whence } m(t_1 - t) = 2\pi \quad \text{Or, } t_1 - t = \frac{2\pi}{m}$$

which gives the interval between consecutive formations of two identical curves.

(e) Vibrations of commensurate frequencies :

(i) Let the two vibrations of frequencies in the ratio of 1 : 2 differing in phase by δ and acting in the same plane be impressed on a particle. Then we can write the component displacements as

$$x = a \sin nt$$

$$y = b \sin (2nt + \delta)$$

$$\text{Then } y = b \sin 2nt \cos \delta + b \cos 2nt \sin \delta$$

$$= 2b \sin nt \sqrt{1 - \sin^2 nt} \cos \delta + b(1 - 2\sin^2 nt) \sin \delta$$

$$= 2b \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} \cos \delta + b \left(1 - \frac{2x^2}{a^2}\right) \sin \delta$$

$$\text{whence } \frac{4b^2 x^2}{a^2} \left(1 - \frac{x^2}{a^2}\right) \cos^2 \delta = y^2 + b^2 \left(1 - \frac{2x^2}{a^2}\right)^2 \sin^2 \delta$$

$$- 2yb \left(1 - \frac{2x^2}{a^2}\right) \sin \delta \quad \dots \quad (14)$$

If the vibrations differ in phase by $\delta = \frac{\pi}{2}$, then we obtain

$$\left(\frac{2x^2}{a^2} + \frac{y}{b} - 1\right)^2 = 0 \quad \dots \quad (15)$$

Equation (14) is one of 4th degree representing a curve having generally two loops. If the vibrations differ by a phase $\frac{\pi}{2}$ the orbit is then two coincident parabolas as represented by the equation (15). If the phase δ changes with slight change in frequencies, the shape of the loop will also change gradually.

(ii) If the frequencies of the vibrations are in the ratio of 1 : 3, then we can write, when the vibrations differ in phase by δ

$$x = a \sin nt$$

$$y = b \sin (3nt + \delta)$$

$$\begin{aligned}
 \therefore y &= b \sin 3nt \cos \delta + b \cos 3nt \sin \delta \\
 &= b(3 \sin nt - 4 \sin^3 nt) \cos \delta + b(4 \cos^3 nt - 3 \cos nt) \sin \delta \\
 &= \left(\frac{3x}{a} - \frac{4x^3}{a^3}\right) b \cos \delta + \left\{4\left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{2}} - 3\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}\right\} b \sin \delta \\
 &= \left(\frac{3x}{a} - \frac{4x^3}{a^3}\right) b \cos \delta + \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \left(1 - \frac{4x^2}{a^2}\right) b \sin \delta
 \end{aligned}$$

Transposing and squaring

$$\left(1 - \frac{x^2}{a^2}\right) \left(1 - \frac{4x^2}{a^2}\right)^2 b^2 \sin^2 \delta = \left\{y - \left(\frac{3x}{a} - \frac{4x^3}{a^3}\right) b \cos \delta\right\}^2 \dots (16)$$

If phase difference $\delta = 0$, we obtain

$$\left(\frac{y}{b} - \frac{3x}{a} + \frac{4x^3}{a^3}\right)^2 = 0 \dots (17)$$

The above represents two coincident cubic curves. If $\delta = \frac{\pi}{2}$, we obtain

$$\left(1 - \frac{x^2}{a^2}\right) \left(1 - \frac{4x^2}{a^2}\right)^2 - \frac{y^2}{b^2} = 0 \dots (16)$$

which is an equation of the sixth degree giving an orbit of three loops. In general if the ratio of the frequencies is N , the curve will have N loops.

If δ in the expression (16) gradually changes, then the shape of the loop also will change.

In those cases where the ratio of frequencies exceed 1 : 3, it is more convenient to find the curve by graphical methods as analytical methods become very cumbersome.

(4) Lissajous' Figures :

The figures formed by two vibrations at right angles to each other are known as Lissajous' Figures. These figures are of importance in sound. With their help, the equality of or a slight difference between the frequencies of two sounding bodies can be tested.

Lissajous used an apparatus known as Vibroscope to study those figures. It is a microscope whose objective O is detached and fixed with the prong A of the tuning fork T_1 which vibrates in a plane perpendicular to axis of the microscope which is, let us suppose, horizontal and in the plane of the paper. Let

T_2 be another fork which can vibrate about an axis perpendicular to the plane of the paper. Suppose both the forks are at rest in the beginning, and a white dot P on T_2 is focussed. When T_1 vibrates, the image of P will appear to be a line perpendicular to the plane of the paper. If now T_2 begins to

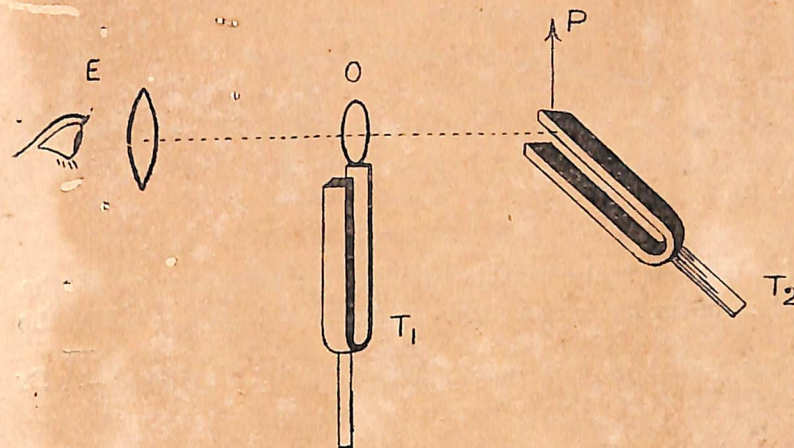


Fig. 3

vibrate, the image of P will have another perpendicular vibration in the vertical line of a frequency equal to that of T_2 . Hence, due to superposition of two rectilinear vibrations, the image will appear to be a pattern generally elliptical if N_1 and N_2 the frequencies of T_1 and T_2 are equal. If N_2 is slightly greater than N_1 , the pattern will gradually change and will repeat after an interval of time

$$= \frac{2\pi}{m} = \frac{2\pi}{2\pi(N_2 - N_1)} = \frac{1}{N_2 - N_1}.$$

Lissajous' figures can also be clearly demonstrated on a screen by reflecting a ray of light successively from two mirrors attached to a prong each of two tuning forks vibrating in planes perpendicular to each other.

Lissajous' figures may be obtained by Blackburn's pendulum when vibrations are of low frequencies. This pendulum consists of a weight D suspended by a thread CD which is attached to another thread ACB fixed at points A and B . The pendulum can vibrate

in the plane of the figure with a time period $T_1 = 2\pi \sqrt{\frac{l_1}{g}}$ where

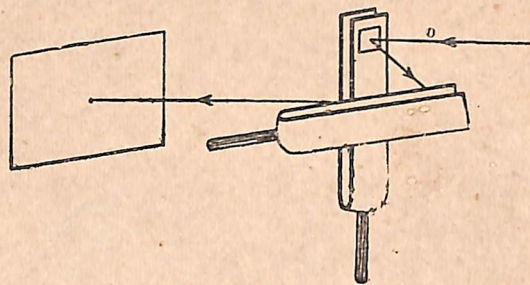


Fig. 4

$l_1 = CD$. It is also capable of vibrating perpendicular to the plane of the figure as a simple pendulum of length $l_2 = ED$, the time period in this case being given by $T_2 = 2\pi \sqrt{\frac{l_2}{g}}$. By

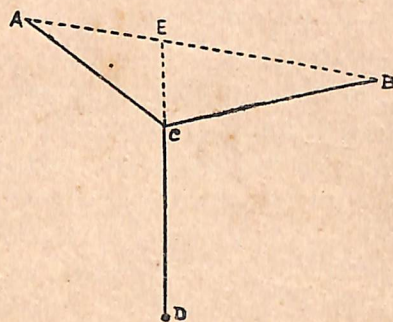


Fig. 5

changing the positions of A and B , this length l_2 can be suitably changed. A record of the resultant of the two vibrations may be obtained by using a funnel at D from which fine sand drops on to a piece of paper and gives Lissajous' figures.

CHAPTER II

DAMPED MOTION

1. In the previous chapter we discussed free simple harmonic motion. This is an ideal thing and is not observed unless some energy is supplied to the vibrating body at a constant rate. In all vibrations observed in nature, e.g., those of a pendulum or of a string etc., the amplitude gradually diminishes and becomes imperceptible after some lapse of time. We may then conclude that there is a damping force on the vibrating particle and this may be due to the viscosity of the medium, or other frictional forces. For small velocity, we may take the damping or frictional force as proportional to the instantaneous velocity of the body.

2. Suppose a particle of mass m is capable of moving along a particular axis OX and while in motion, is subject to a restoring force proportional to the distance from a fixed point on the axis and a frictional force proportional to the velocity. Then the equation of motion of such a particle can be written as

$$m \frac{d^2 x}{dt^2} = -sx - k \frac{dx}{dt}$$

when s is the stiffness constant and k may be called the resistance coefficient which denotes frictional force per unit velocity, x being the displacement of the particle from the fixed point at any instant t .

Re-arranging

$$\frac{d^2 x}{dt^2} + \frac{k}{m} \cdot \frac{dx}{dt} + \frac{s}{m} x = 0$$

Writing $n^2 = \frac{s}{m}$ and $2b = \frac{k}{m}$ we have

$$\frac{d^2 x}{dt^2} + 2b \cdot \frac{dx}{dt} + n^2 x = 0 \quad \dots \quad (1)$$

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This is an equation of the second order. Let $x = e^{\alpha t}$ be a particular solution of the equation. Then

$$\frac{dx}{dt} = \alpha e^{\alpha t}, \quad \frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

After substitution in (1) we get

$$\alpha^2 + 2b\alpha + n^2 = 0$$

$$\therefore \alpha = -b \pm \sqrt{b^2 - n^2}$$

Thus the general solution of equation (1) can be written as

$$\begin{aligned} x &= A_1 e^{(-b + \sqrt{b^2 - n^2})t} + A_2 e^{(-b - \sqrt{b^2 - n^2})t} \\ &= e^{-bt} (A_1 e^{\sqrt{b^2 - n^2}t} + A_2 e^{-\sqrt{b^2 - n^2}t}) \quad \dots (2) \end{aligned}$$

when A_1 and A_2 are two constants whose values can be determined from initial conditions.

Case (1) Damping force large so that $b > n$

In the expression (2), let us put $b_1 = \sqrt{b^2 - n^2}$;

$$\text{then } x = e^{-bt} (A_1 e^{b_1 t} + A_2 e^{-b_1 t})$$

Differentiating

$$\frac{dx}{dt} = -be^{-bt} (A_1 e^{b_1 t} + A_2 e^{-b_1 t}) + e^{-bt} (b_1 A_1 e^{b_1 t} - b_1 A_2 e^{-b_1 t})$$

If $x = a$ and $\frac{dx}{dt} = v_0$ at $t = 0$, we have

$$A_1 + A_2 = a$$

$$-b(A_1 + A_2) + b_1(A_1 - A_2) = v_0$$

$$\text{or, } -ba + b_1(A_1 - A_2) = v_0$$

$$\text{whence } A_1 - A_2 = \frac{v_0 + ba}{b_1}$$

$$\text{Thus we get } A_1 = \frac{a}{2} \left(1 + \frac{b + v_0/a}{b_1} \right)$$

$$\text{and } A_2 = \frac{a}{2} \left(1 - \frac{b + v_0/a}{b_1} \right)$$

And the solution for x , substituting the value of b , is

$$x = \frac{a}{2} e^{-bt} \left\{ \left(1 + \frac{b + v_0/a}{\sqrt{b^2 - n^2}} \right) e^{\sqrt{b^2 - n^2}t} + \left(1 - \frac{b + v_0/a}{\sqrt{b^2 - n^2}} \right) e^{-\sqrt{b^2 - n^2}t} \right\} \quad \dots (3)$$

If the particle is displaced to a and then released so that $\frac{dx}{dt} = v_0 = 0$ at $t = 0$, the damped motion will be as represented in fig. 6.

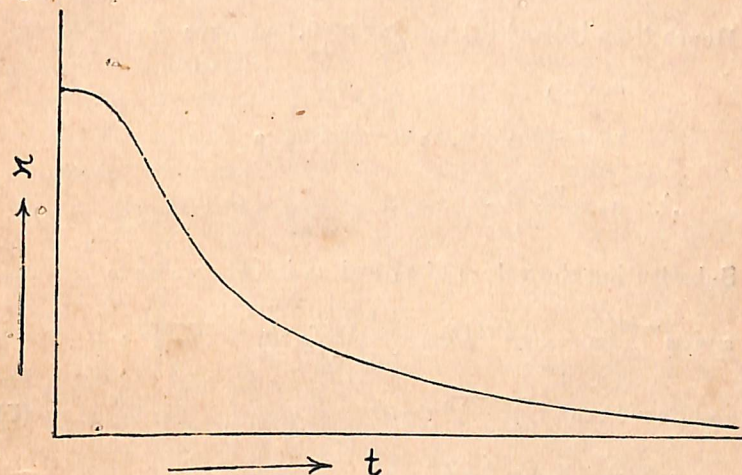


Fig. 6

Case (2) Let b be small so that $b < n$, then

$$\sqrt{b^2 - n^2} = i \sqrt{n^2 - b^2}$$

Hence equation (2) reduces to

$$\begin{aligned} x &= e^{-bt} (A_1 e^{i \sqrt{n^2 - b^2}t} + A_2 e^{-i \sqrt{n^2 - b^2}t}) \\ &= e^{-bt} \{ (A_1 + A_2) \cos \sqrt{n^2 - b^2}t + i(A_1 - A_2) \sin \sqrt{n^2 - b^2}t \} \end{aligned}$$

Now A_1 and A_2 may both contain real as well as imaginary parts. Let a_1, a_2 be the real parts of the coefficients of $\cos \sqrt{n^2 - b^2}t$ and $\sin \sqrt{n^2 - b^2}t$ respectively.

Then,

$$x = e^{-bt} (a_1 \cos \sqrt{n^2 - b^2}t + a_2 \sin \sqrt{n^2 - b^2}t) \quad \dots (4)$$

$$= e^{-bt} (a_1 \cos n_1 t + a_2 \sin n_1 t) \quad \dots (5)$$

$$\text{writing } n_1 = \sqrt{n^2 - b^2}$$

The velocity at any instant t is

$$\frac{dx}{dt} = -be^{-bt}(a_1 \cos n_1 t + a_2 \sin n_1 t) + e^{-bt}(-n_1 a_1 \sin n_1 t + n_1 a_2 \cos n_1 t) \quad \dots (6)$$

Let the displacement and velocity of the particle at $t=0$ be a and v_0 respectively.

Hence from (5) and (6), putting $t=0$, we have

$$a = a_1$$

$$v_0 = -ba_1 + n_1 a_2 = -ba + n_1 a_2$$

$$\therefore a_2 = \frac{v_0 + ba}{n_1}$$

Substituting the values of a_1 and a_2 in (4) we have

$$x = ae^{-bt} \left(\cos \sqrt{n^2 - b^2} t + \frac{b + v_0/a}{\sqrt{n^2 - b^2}} \sin \sqrt{n^2 - b^2} t \right) = Re^{-bt} \cos(\sqrt{n^2 - b^2} t - \theta) \quad \dots (7)$$

$$\text{where } R \cos \theta = a$$

$$R \sin \theta = \frac{ab + v_0}{\sqrt{n^2 - b^2}}$$

$$\text{so that } R = \sqrt{\frac{a^2 n^2 + v_0^2 + 2abv_0}{n^2 - b^2}}$$

$$\text{and } \tan \theta = \frac{b + v_0/a}{\sqrt{n^2 - b^2}}$$

If the body is displaced to a and then released, we must have $\frac{dx}{dt} = v_0 = 0$ at $t=0$ and then we get

$$x = \frac{ane^{-bt}}{\sqrt{n^2 - b^2}} \cos(\sqrt{n^2 - b^2} t - \theta') \quad \dots (8)$$

$$\text{where } \tan \theta' = \frac{b}{\sqrt{n^2 - b^2}}$$

Thus the motion expressed by (7) or (8) is a damped oscillatory motion, the amplitude decreasing exponentially with time.

The time period $T = \frac{2\pi}{\sqrt{n^2 - b^2}}$ is slightly greater than the time period for free natural vibration which is $\frac{2\pi}{n}$; but since $n^2 = \frac{s}{m}$ and $b^2 = \frac{k^2}{4m^2}$, b^2 is a term of smallness of second order in comparison with n^2 and can be in most cases neglected. Thus time period of oscillation is very slightly affected by damping of ordinary magnitude. The damped oscillatory motion as represented by equation (7) is illustrated by fig. 7.

Let x_0, x_1, x_2, x_3 etc., be the maximum displacements of the system in both directions at times given by $n_1 t - \theta = 0, \pi, 2\pi, 3\pi$ etc., respectively where $n_1 = \sqrt{n^2 - b^2}$. Then

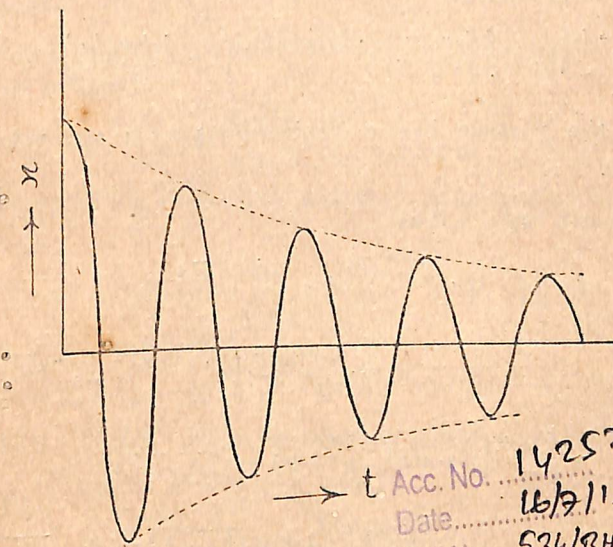


Fig. 7

$$\begin{aligned} x_0 &= Re^{-\frac{b\theta}{n_1}} \\ x_1 &= -Re^{-\frac{b(\theta+\pi)}{n_1}} \\ x_2 &= Re^{-\frac{b(\theta+2\pi)}{n_1}} \\ x_3 &= -Re^{-\frac{b(\theta+3\pi)}{n_1}} \end{aligned}$$

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Thus, neglecting signs of displacements

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \dots = e^{\frac{b\pi}{n}} = e^{b \cdot \frac{T}{2}}$$

$$\text{Hence } b = \frac{2}{T} \cdot \log \frac{x_0}{x_1} = \frac{2}{T} \cdot \log \frac{x_1}{x_2} \dots \text{etc.}$$

Thus the damping coefficient b can be found out from an experimental measurement of consecutive amplitudes.

Case (3). If $b \rightarrow n$, then this is a transitional case when the damped dead beat motion changes to damped oscillatory vibrations.

We have from (3)

$$x = \frac{a}{2} e^{-bt} \left\{ \left(1 + \frac{b+v_0/a}{\sqrt{b^2-n^2}} \right) e^{\sqrt{b^2-n^2}t} + \left(1 - \frac{b+v_0/a}{\sqrt{b^2-n^2}} \right) e^{-\sqrt{b^2-n^2}t} \right\}$$

when the particle has a displacement $x=a$ and velocity v_0 at $t=0$.

We can rewrite it as

$$\begin{aligned} x &= \frac{a}{2} e^{-bt} \left\{ e^{\sqrt{b^2-n^2}t} + e^{-\sqrt{b^2-n^2}t} \right. \\ &\quad \left. + \frac{b+v_0/a}{\sqrt{b^2-n^2}} \left(e^{\sqrt{b^2-n^2}t} - e^{-\sqrt{b^2-n^2}t} \right) \right\} \\ &= \frac{a}{2} e^{-bt} \left\{ 1 + \sqrt{b^2-n^2}t + 1 - \sqrt{b^2-n^2}t \right. \\ &\quad \left. + \frac{b+v_0/a}{\sqrt{b^2-n^2}} \left(1 + \sqrt{b^2-n^2}t - 1 + \sqrt{b^2-n^2}t \right) \right\} \end{aligned}$$

expanding and neglecting higher order terms since $b \rightarrow n$.

$$\text{Hence } x = \frac{a}{2} e^{-bt} \{ 2 + 2(b+v_0/a)t \}$$

Thus the general solution for critically damped motion can be written as

$$x = e^{-bt} (A + Bt)$$

2. An application of the theory of damped oscillations :

The theory of damped oscillation may be utilised in finding

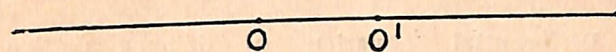


Fig. 8

the true resting point of the pointer of an oscillating balance (method of oscillations).

Let O be the true resting point of the pointer of a balance. Let the displacements be measured from the arbitrary mark O' . Let $OO' = \alpha$. Starting from the right hand side let x_1, x_2, x_3, x_4, x_5 be the consecutive maximum displacements from O' , x_1, x_3, x_5 being taken on the right hand side of O' and x_2, x_4 on the left hand side.

Since it is a case of damped oscillation the displacement from the resting point is

$$\begin{aligned} x &= ae^{-bt} \cos(n_1 t - \theta) \\ &= ae^{-bt} \cos n_1 t \text{ neglecting } \theta \end{aligned}$$

Hence $x_1 + \alpha = a$, time being reckoned from the instant of measuring x_1

$$\text{Then } x_3 + \alpha = ae^{-bT}$$

$$\text{and } x_5 + \alpha = ae^{-2bT}$$

$$\begin{aligned} \therefore x_1 + x_3 + x_5 &= a \{ 1 + e^{-bT} + e^{-2bT} \} - 3\alpha \\ &= a \{ 1 + 1 - bT + 1 - 2bT \} - 3\alpha, \text{ neglecting higher order terms.} \end{aligned}$$

$$\therefore \frac{x_1 + x_3 + x_5}{3} = a \{ 1 - bT \} - \alpha$$

$$\text{Similarly, } x_2 - \alpha = ae^{-b \frac{T}{2}}$$

$$\text{and } x_4 - \alpha = ae^{-8b \frac{T}{2}}$$

$$\begin{aligned} \therefore x_2 + x_4 &= a \left\{ e^{-b \frac{T}{2}} + e^{-4b \frac{T}{2}} \right\} + 2\alpha \\ &= a \left\{ 1 - \frac{bT}{2} + 1 - \frac{4bT}{2} \right\} + 2\alpha \end{aligned}$$

$$\therefore \frac{x_2 + x_4}{2} = a \{ 1 - bT \} + \alpha$$

Hence $\frac{x_2 + x_4}{2} - \frac{x_1 + x_3 + x_5}{3} = 2\alpha$

Thus OO' and hence the true position of rest O can be found.

3. Differential equation of damped motion from consideration of the energy of the system :

The differential equation of damped motion can be obtained from the consideration of the energy of the system. Let the displacement at any instant be x ; if the restoring force is proportional to x , the potential energy of the particle at the instant is

$$\int_0^x sx \, dx = \frac{sx^2}{2}$$

The kinetic energy at the same instant is $\frac{1}{2}m \left(\frac{dx}{dt}\right)^2$. If the particle further describes an element of displacement δx , then the loss of kinetic and potential energy of the particle will be equal to the work done against the frictional force.

Or, $-d\left\{\frac{sx^2}{2} + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2\right\} = k \cdot \frac{dx}{dt} \cdot \delta x$

Or, $\frac{dx}{dt} \left\{ m \frac{d^2x}{dt^2} + k \cdot \frac{dx}{dt} + sx \right\} = 0$

Since $\frac{dx}{dt} \neq 0$ for all values of t

$$m \frac{d^2x}{dt^2} + k \cdot \frac{dx}{dt} + sx = 0$$

FORCED VIBRATION AND RESONANCE

1. Motion due to a periodic force : In the previous chapters, we treated "free" vibrations of a system. Let us now consider the case when a periodic force of constant frequency and amplitude acts on it. Initial transient vibrations will be set up which will soon die down and the system will settle down to a sustained "forced vibration" of the same frequency as that of the periodic force. We can cite many common examples of forced vibration, *e. g.*, that of a loudspeaker cone, a gramophone sound-box, a stretched wire under tension actuated by the vibrations of a tuning fork etc.

Let an external simple harmonic force $F \sin pt$ act on a mass m which when displaced is subjected to a restoring force proportional to displacement and a frictional force proportional to velocity. If the system has one degree of freedom along a given axis OX , then its motion will be given by

$$m \frac{d^2x}{dt^2} = -sx - k \frac{dx}{dt} + F \sin pt$$

where s is the stiffness constant, k the resistance constant, F the amplitude and p the angular frequency of the simple harmonic force.

Re-arranging the equation, we have

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + n^2x = f \sin pt$$

where $2b = \frac{k}{m}$, $n^2 = s/m$ and $f = F/m$... (1)

Let a value of $x = x_1$ be found out which satisfies the equation

$$\frac{d^2x_1}{dt^2} + 2b \frac{dx_1}{dt} + n^2x_1 = 0$$

and also let another value of $x = x_2$ be the particular solution of the equation, so that

$$\frac{d^2x_2}{dt^2} + 2b \frac{dx_2}{dt} + n^2x_2 = f \sin pt \quad \dots (2)$$

Then

$$\frac{d^2}{dt^2}(x_1 + x_2) + 2b \frac{d}{dt}(x_1 + x_2) + n^2(x_1 + x_2) = f \sin pt$$

The solution x_1 is the same as in the case of resisted oscillation and is given by

$$x_1 = A_1 e^{-bt} \sin(\sqrt{n^2 - b^2}t + \theta) \quad \dots (3)$$

where A and θ are constants.

To find x_2 , let us take as solution $x_2 = A \sin(pt - \alpha)$. This supposition we can make on the ground that the system will ultimately vibrate with the same frequency as that of the impressed sustained harmonic force.

Since $x_2 = A \sin(pt - \alpha)$

$$\frac{dx_2}{dt} = Ap \cos(pt - \alpha)$$

$$\text{and } \frac{d^2x_2}{dt^2} = -Ap^2 \sin(pt - \alpha)$$

Substituting these values of $\frac{dx_2}{dt}$ and $\frac{d^2x_2}{dt^2}$ in (2)

$$-Ap^2 \sin(pt - \alpha) + 2bAp \cos(pt - \alpha) + n^2A \sin(pt - \alpha) = f \sin pt$$

$$\text{Or, } A(n^2 - p^2) \sin(pt - \alpha) + 2bAp \cos(pt - \alpha)$$

$$= f \sin(pt - \alpha + \alpha)$$

$$= f \sin(pt - \alpha) \cos \alpha + f \cos(pt - \alpha) \sin \alpha$$

Since the above equation is true for all values of t , we can equate the coefficients of $\sin(pt - \alpha)$ and $\cos(pt - \alpha)$ from both sides.

Hence

$$f \cos \alpha = A(n^2 - p^2)$$

$$f \sin \alpha = 2bAp$$

$$\left. \begin{array}{l} f \cos \alpha = A(n^2 - p^2) \\ f \sin \alpha = 2bAp \end{array} \right\} \dots (4)$$

$\therefore$ Squaring and adding

$$f^2 = A^2 \{(n^2 - p^2)^2 + 4b^2 p^2\}$$

$$\text{whence } A = \frac{f}{\sqrt{(n^2 - p^2)^2 + 4b^2 p^2}} \quad \dots (5)$$

Also from (4)

$$\tan \alpha = \frac{2bp}{n^2 - p^2}$$

Thus the complete solution for x is

$$x = x_1 + x_2 = A_1 e^{-bt} \sin(\sqrt{n^2 - b^2}t + \theta) + \frac{f}{\sqrt{(n^2 - p^2)^2 + 4b^2 p^2}} \sin(pt - \alpha) \quad \dots (6)$$

$$\text{where } \tan \alpha = \frac{2bp}{n^2 - p^2}$$

The first part of the solution for x , i. e., x_1 represents natural vibrations set up in the damped system by the harmonic force at the start. These vibrations, however, become negligible very soon as the amplitude diminishes exponentially with time. If damping is very small, the natural vibrations will persist for a longer time. The resultant vibration x at any instant is the sum of the natural vibration represented by x_1 and the forced sustained vibration represented by x_2 . After a lapse of time when x_1 becomes negligible, we can write $x = A \sin(pt - \alpha)$ which represents the sustained forced vibration.

If $\sqrt{n^2 - b^2}$ and p are nearly equal, at the initial stage, the natural vibration will interfere with the forced vibration and produce beats. These beats are transient, as natural vibrations become imperceptible after a short interval of time.

2. Response and resonance :

Writing $x = A \sin(pt - \alpha)$ for the steady forced vibration, we

$$\text{have } \frac{dx}{dt} = Ap \cos(pt - \alpha)$$

Hence kinetic energy at any instant

$$\frac{1}{2}m \left(\frac{dx}{dt} \right)^2 = \frac{1}{2}mA^2 p^2 \cos^2(pt - \alpha)$$

Since the motion is a steady harmonic motion, the maximum kinetic energy of the system is its total energy at any instant.

Hence energy of the system

$$\begin{aligned} &= \frac{1}{2}mA^2 p^2 \\ &= \frac{1}{2}mp^2 \cdot \frac{f^2}{(n^2 - p^2)^2 + 4b^2 p^2} \\ &= \frac{\frac{1}{2}mf^2}{n^2 \left(\frac{n^2 - p^2}{p} \right)^2 + 4b^2} \end{aligned}$$

If $p=n$, the energy of the system is maximum for any given value of b . Thus when the frequency of the driven coincides with the natural frequency of the driver (without damping), the energy of the driven system is maximum. This phenomenon is known as *velocity resonance* or *energy resonance* or simply *resonance*. Moreover, the decrease of energy upon the lack of coincidence is the same for a given ratio of frequencies whether the frequency of the impressed force is too great or too small. It means the energy of the system is the same whether the angular frequency p of the impressed force is s times n or $\frac{1}{s}$ of n . Writing $\frac{n}{p} - \frac{p}{n} = \Delta$, the kinetic energy at any mistuning or the energy of response can be written as

$$\left. \begin{aligned} E &= \frac{\frac{1}{2}mf^2}{n^2\Delta^2 + 4b^2} \\ \text{and energy at resonance } E_m &= \frac{\frac{1}{2}mf^2}{4b^2} \end{aligned} \right\} \dots (7)$$

Thus for a particular value of the amplitude of the impressed force, the kinetic energy is greater for a smaller value of b .

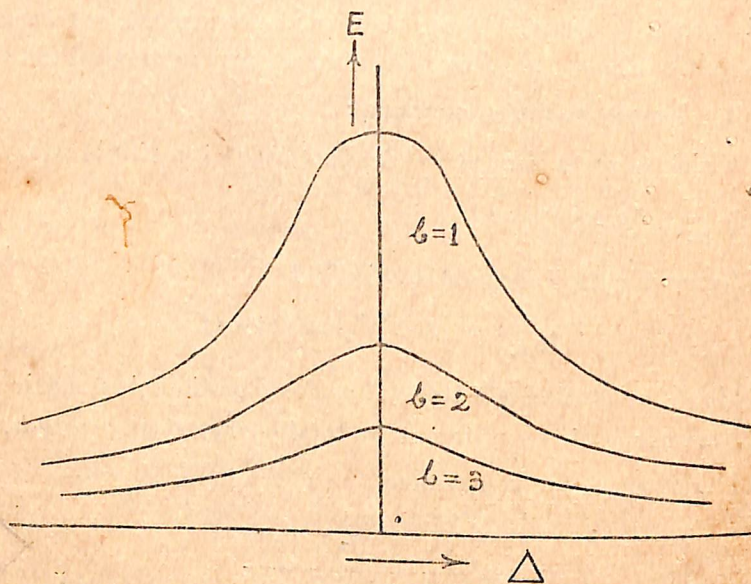


Fig. 9

Again if we plot E with Δ we shall get response curves for different values of b as given in the figure above.

We have from (7)

$$\begin{aligned} \frac{E}{E_m} &= \frac{4b^2}{n^2\Delta^2 + 4b^2} \\ &= \frac{1}{1 + \frac{n^2\Delta^2}{4b^2}} \dots (8) \end{aligned}$$

For all values of damping constant b , $\frac{E}{E_m} = 1$ at $\Delta = 0$, i.e., at $n=p$ which is the condition for resonance; again $\frac{E}{E_m}$ will be zero at very large values of Δ i.e., when the frequency of the driver is too large or too small compared to the natural frequency of the driven. But if b is negligible and n large $\frac{E}{E_m}$ will be negligible even at moderate values of Δ , as the denominator in the expression (8) then becomes very large. Thus for vibrations with b small and n large, the response E at any mistuning Δ will be very small in comparison with energy at resonance and hence resonance will be sharp. For the hypothetical case $b \rightarrow 0$, the response for a slight mistuning is almost zero as the denominator in (8) tends to infinity. Again if b is very large $\frac{E}{E_m}$ remains virtually of the same value equal to unity for moderate values of Δ and the response is very flat. The observations are illustrated in fig. 9.

The sharpness of resonance is sometimes quantitatively defined as the reciprocal of Δ at which energy of response is half of that at resonance

Since $E_m = \frac{1}{2} \frac{mf^2}{4b^2}$, the value of Δ at which energy of response is half of that at resonance is given by

$$E = \frac{\frac{1}{2}mf^2}{n^2\Delta^2 + 4b^2} = \frac{1}{2}E_m = \frac{1}{2} \cdot \frac{\frac{1}{2}mf^2}{4b^2}$$

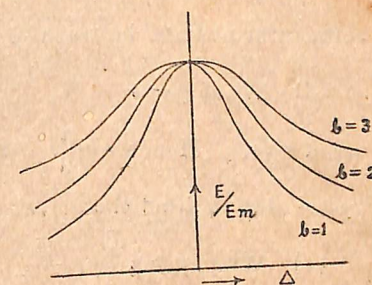


Fig. 10

whence $n^2 \Delta^2 + 4b^2 = 8b^2$

$$\therefore \frac{1}{\Delta} = \pm \frac{n}{2b} = s$$

The sharpness of resonance is then proportional directly to the natural frequency of the system and inversely to the damping constant b .

3. Phase of the driven system with respect to that of the driving force :

We have

$$\tan \alpha = \frac{2bp}{n^2 - p^2}$$

$$\text{and } \sin \alpha = \frac{2bAp}{f} \quad \text{from (4)}$$

From this value of $\sin \alpha$, we see that it is always positive which shows that α lies between 0 and π .

Suppose the angular frequency of the impressed force is increased gradually from 0 to ∞ .

(1) When $p=0$, $\tan \alpha=0$. Hence $\alpha=0$

Thus there is no difference of phase between the driven and the driver.

(2) When $p < n$

$\tan \alpha$ is +ve; it means that the difference of phase has a value intermediate between 0 and $\frac{\pi}{2}$.

(3) When $p=n$

$\tan \alpha = \infty$. Hence $\alpha = \frac{\pi}{2}$. Thus at resonance, the driven system lags behind the driver by an angle $\frac{\pi}{2}$.

(4) When $p > n$

In this case, $\tan \alpha$ is -ve, hence α is an angle in the 2nd quadrant or, $\frac{\pi}{2} < \alpha < \pi$.

Hence as $p \rightarrow \infty$ $\tan \alpha \rightarrow 0$ i.e., α in this case is π .

Thus for all values of p , α lies between 0 and π , being equal to $\frac{\pi}{2}$ at resonance.

Again since

$$\alpha = \tan^{-1} \frac{2bp}{(n^2 - p^2)}$$

$$\frac{d\alpha}{dp} = \frac{2b(n^2 + p^2)}{(n^2 - p^2)^2 + 4b^2 p^2}$$

Hence, when n is equal to p , $\frac{d\alpha}{dp} = \frac{1}{b}$.

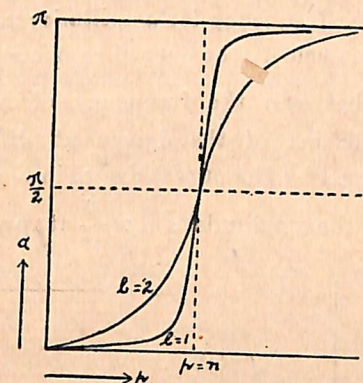


Fig. 11

Thus smaller the value of b , greater the rate of change of phase angle near resonance frequency. Relations of α with p are given in fig. 11.

4. Amplitude of forced vibration :

Let us find condition when amplitude of vibration of the driven system is greatest. We have

$$A = \frac{f}{\sqrt{(n^2 - p^2)^2 + 4b^2 p^2}}$$

$$\begin{aligned} \therefore \frac{dA}{dp} &= f \left[\frac{-\frac{1}{2} \{ 2(n^2 - p^2) \times -2p + 4b^2 \cdot 2p \}}{\{(n^2 - p^2)^2 + 4b^2 p^2\}^{3/2}} \right] \\ &= f \cdot p \cdot \frac{2(n^2 - p^2) - 4b^2}{\{(n^2 - p^2)^2 + 4b^2 p^2\}^{3/2}} \end{aligned}$$

$$\frac{dA}{dp} = 0, \text{ when } p = \infty \text{ and } (n^2 - p^2) - 2b^2 = 0.$$

The former condition gives amplitude zero. Hence the condition for maximum amplitude is

$$n^2 - p^2 = 2b^2$$

$$\text{Or, } p^2 = n^2 - 2b^2$$

If b is very small

$$p = n \left(1 - \frac{2b^2}{n^2} \right)^{\frac{1}{2}} \quad \dots \dots \dots (9)$$

$$\approx n - \frac{b^2}{n}$$

Thus the angular frequency for *amplitude resonance* is slightly smaller than that at velocity or energy resonance.

The relation between the frequency of the driver and the amplitude of oscillation of the driven at different damping constants is explained by the curves drawn in figure 12. For all values of damping the amplitude is $A = \frac{f}{n^2}$ at $p=0$. Now if b is

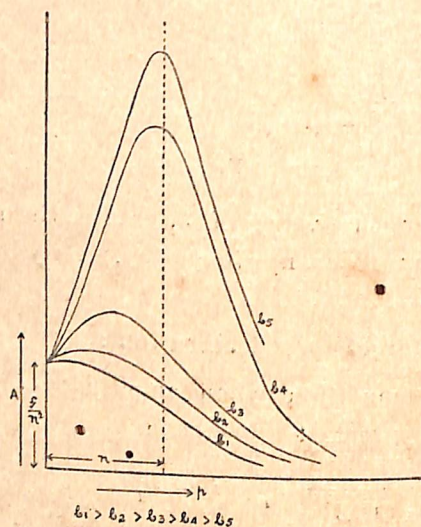


Fig. 12

very large, the increase in $2bp$ as p increases towards n is larger than the decrease in the value of $n^2 - p^2$ and thus the denominator in the expression for A increases with increase of p . Hence in these cases A will be maximum at $p=0$. If b is comparatively small, maximum amplitude occurs at $p^2 = n^2 - 2b^2$ i. e., at an

angular frequency smaller than the resonance angular frequency by an amount determined by the value of b . For a very small value of b , p is almost equal to n . If damping is absent, amplitude becomes infinite at $p=n$, the resonance frequency.

5. Power supplied by the driving force :

Since energy is dissipated in each cycle due to frictional force, this loss must be made up by the energy of the driving force to maintain the steady forced vibration.

Suppose, at any instant, the force $F \sin pt$ moves through a distance δx in time δt . Then work done is $F \sin pt. \delta x$.

Hence rate of work done

$$\begin{aligned} &= \frac{1}{T} \int_0^T F \sin pt. \frac{dx}{dt} dt \\ &= \frac{1}{T} \int_0^T F \sin pt \times pA \cos(pt - \alpha). dt \quad [\because x = A \sin(pt - \alpha)] \\ &= \frac{1}{T} \int_0^T FpA (\sin pt \cos pt \cos \alpha + \sin^2 pt \sin \alpha). dt \\ &= \frac{1}{T} \cdot FpA \cdot \frac{T}{2} \cdot \sin \alpha \\ &[\because \int_0^T \sin pt \cos pt dt = 0. \quad \int_0^T \sin^2 pt. dt = T/2] \\ &= \frac{FpA \cdot \sin \alpha}{2} \end{aligned}$$

Also work done against frictional resistance for the displacement δx is $k \frac{dx}{dt} \delta x$. Hence rate of work done against frictional

$$\begin{aligned} \text{force} &= \frac{1}{T} \int_0^T k. \frac{dx}{dt} \cdot \frac{dx}{dt} dt = \frac{1}{T} \int_0^T k A^2 p^2 \cos^2(pt - \alpha). dt \\ &= \frac{1}{T} \cdot k \cdot A^2 P^2 \cdot T/2 = \frac{kA^2 p^2}{2} \end{aligned}$$

$$\begin{aligned}\text{Now } \frac{Fp}{2} \frac{A \sin \alpha}{f} &= \frac{FpA}{2} \frac{2b}{f} \frac{Ap}{f} \\ &= \frac{FpA}{2} \frac{k}{m} Ap \times \frac{m}{F} = \frac{k}{2} A^2 p^2\end{aligned}$$

Thus the rate of supply of energy by the driving force is equal to the rate of work done against the frictional resistance.

6. Advantages and disadvantages of resonance :

Resonance is both a useful as well as an obnoxious phenomenon. It is very often utilised in finding an unknown frequency (e. g., by a sonometer) or in detecting a particular frequency present in a note consisting of a conglomeration of frequencies (e. g., by Helmholtz Resonator). In these cases sharpness of resonance is of much advantage. Resonance effect is also utilised in Indian stringed instruments like Setar, Esraj, Sarode etc., which have additional strings other than the main ones tuned to the desired notes of the scale.

But sharp resonance in many cases is very undesirable. A microphone, a loudspeaker or any other instrument meant for recording or reproducing music must have a flat response to the range of frequencies meant to reproduce. Resonance to any of the frequencies within the range will mean an undue augmentation of it and will result in distortion. A flat response curve is obtained by making the natural frequency of the system either much lower or much higher than the frequencies to be reproduced and by making damping very large.

7. Experiment on forced vibration and resonance :

Let us take a thin thread fixed at A and B . Let two pendulums CE and DF be attached to positions C and D on the thread. The pendulum CE is a heavy compound pendulum, whereas DF is a simple pendulum whose bob is of a light material like cork. Now oscillations started in CE will force the pendulum DF to oscillate also, and since DF is light, there will be little feed-back of energy from DF to CE so that in this case we can take CE as the driver and DF as the driven.

Let us start with the thread of the pendulum DF short in comparison with the equivalent length of CE . The natural time periods of the two are very different in this case. If oscillations

are started in CE , there are at first transient oscillations in DF which ultimately die down and the pendulum DF settles down to oscillate with the frequency of the driver. The amplitude is

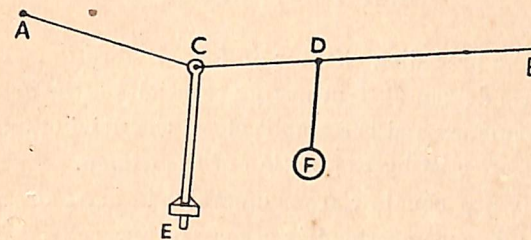


Fig. 13

however very small because of the difference in the natural periods of the pendulums. The oscillations are almost in phase. This should be so, for the phase of the driven is given by $\tan^{-1} \frac{2bp}{n^2 - p^2}$, which is nearly zero in this case as $p \ll n$.

Now let us stop the oscillations of the pendulums, increase the length of DF and start oscillations in CE . We shall observe that the forced vibration in this case is more vigorous and there is some difference of phase between the oscillations of the two. If the natural periods of the driver and the driven are made same, resonance would occur as a result of which the vibrations of DF are most vigorous. The phase difference between the two oscillations in this case must be $\frac{\pi}{2}$, and we shall observe that when CE is in the middle of a vibration, DF is at the end of it and vice versa.

If we make observations with the length of DF very large in comparison with that of CE , we shall still see that DF ultimately oscillates with the period of the driver, but as expected, with a greatly decreased amplitude. The phase angle $\tan^{-1} \frac{2bp}{n^2 - p^2}$ is nearly π here as p is large in comparison with n . As expected we would see the two oscillations taking place almost out of phase. The position of CE and DF will be at the two opposite ends of their displacements simultaneously.

* For further discussion on sharpness of resonance, see Appendix.

CHAPTER IV

COUPLED VIBRATIONS

1. In the preceding chapter on forced vibration, it has been assumed that driven system has no reaction on the driving force, i. e., the frequency and the amplitude of the driving force are not in the least affected by the motion of the driven. But when the two systems are equal or are comparable in mass or inertia, the roles of the driver and the driven may be interchanged. Motion of each system will react on that of the other and the two systems in this case are coupled with each other. This can be illustrated by the following example.

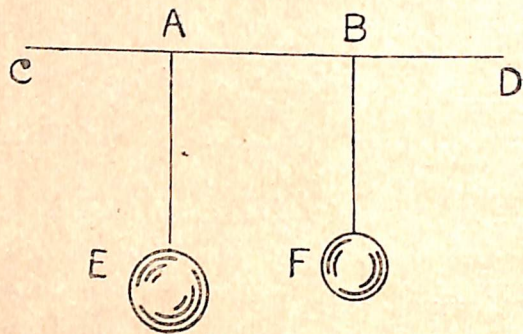


Fig. 14

Suppose AE and BF are two pendulums attached to a thread CD at A and B . If the mass of AE is very large in comparison with that of BF , then the vibration perpendicular to the plane of the paper initiated in BF by that in AE will be what is known as forced vibration. Now suppose both the pendulums are of nearly the same mass and length and motion is started by giving AE an initial displacement. It will be seen that BF soon takes up motion. The amplitude of vibration of BF gradually increases while that of AE steadily falls. Soon AE will come almost to rest. And now the position of the affairs will be reversed; AE will now take up energy from BF ; its vibration will now gradually increase in amplitude while that in BF will fall. This process will be repeated until the energy is dissipated due to friction.

Let m_1 and m_2 be the masses of the two pendulums respectively. Suppose at any instant, x_1 is the displacement of the pendulum of mass m_1 ; then the force on it at that instant in the direction of displacement is $m_1 \frac{d^2 x_1}{dt^2}$ which gives rise to a

reaction at the point of suspension proportional to $-\frac{d^2 x_1}{dt^2}$ parallel to x_1 . Due to mass-acceleration coupling, the force transmitted to the pendulum of mass m_2 must be proportional to $-\frac{d^2 x_1}{dt^2}$ and can be written as $-\mu_2 \frac{d^2 x_1}{dt^2}$ where μ_2 is a constant depending on the extent of coupling. If we neglect damping forces for the present, the equation of motion of m_2 can be written as

$$m_2 \frac{d^2 x_2}{dt^2} + \mu_2 \frac{d^2 x_1}{dt^2} + s_2 x_2 = 0 \quad \dots \quad (1)$$

where s_2 is the stiffness constant of the system and x_2 the displacement of m_2 parallel to that of m_1 . Let the displacements be counted as positive when they are along a particular direction, say, from left to right.

From similar considerations, the force on m_1 due to motion of m_2 must be proportional to $-\frac{d^2 x_2}{dt^2}$ and its motion can be expressed as

$$m_1 \frac{d^2 x_1}{dt^2} + \mu_1 \frac{d^2 x_2}{dt^2} + s_1 x_1 = 0 \quad \dots \quad (2)$$

It can be proved in the following way that $\mu_1 = \mu_2$. Let us multiply equation (2) by $\frac{dx_1}{dt}$ and equation (1) by $\frac{dx_2}{dt}$ when we get from them

$$\frac{d}{dt} \left\{ \frac{1}{2} m_1 \left(\frac{dx_1}{dt} \right)^2 + \frac{1}{2} s_1 x_1^2 \right\} + \mu_1 \frac{d^2 x_2}{dt^2} \cdot \frac{dx_1}{dt} = 0 \quad \dots \quad (3)$$

$$\frac{d}{dt} \left\{ \frac{1}{2} m_2 \left(\frac{dx_2}{dt} \right)^2 + \frac{1}{2} s_2 x_2^2 \right\} + \mu_2 \frac{d^2 x_1}{dt^2} \cdot \frac{dx_2}{dt} = 0 \quad \dots \quad (4)$$

From (3) & (4), after integration and addition we obtain,

$$\left\{ \frac{1}{2} m_1 \left(\frac{dx_1}{dt} \right)^2 + \frac{1}{2} s_1 x_1^2 \right\} + \frac{\mu_1}{\mu_2} \left\{ \frac{1}{2} m_2 \left(\frac{dx_2}{dt} \right)^2 + \frac{1}{2} s_2 x_2^2 \right\} + \mu_1 \frac{dx_1}{dt} \cdot \frac{dx_2}{dt} = C \quad \dots \quad (5)$$

where C is the constant of integration.

Let us suppose the motion is started by giving m_1 an initial velocity u_0 at $t=0$.

$$\text{Then at } t=0, \frac{1}{2} m_1 \left(\frac{dx_1}{dt} \right)^2 = \frac{1}{2} m_1 u_0^2, x_1 = x_2 = 0, \frac{dx_2}{dt} = 0.$$

Thus the constant C is the total energy of the coupled system and equation (5) is the energy equation. But energy at any instant must consist of the kinetic energies of the two masses together with other energies. Hence the term $\frac{\mu_1}{\mu_2} \frac{1}{2} m_2 \left(\frac{dx_2}{dt} \right)^2$ must be independent of μ_1 and μ_2 , that is, μ_1 must be equal to μ_2 . Writing in equations (2) & (1) $\mu_1 = \mu_2 = \mu$ we get the equations of motion of this coupled system as

$$m_1 \frac{d^2 x_1}{dt^2} + \mu \frac{d^2 x_2}{dt^2} + s_1 x_1 = 0 \quad \dots \quad (6)$$

$$m_2 \frac{d^2 x_2}{dt^2} + \mu \frac{d^2 x_1}{dt^2} + s_2 x_2 = 0 \quad \dots \quad (7)$$

Since the motions are periodic, we can write as a trial solution for (6)

$$x_1 = A e^{i p t}$$

where A is a constant.

Substituting the values of $\frac{d^2 x_1}{dt^2}$ in (7) we get

$$m_2 \frac{d^2 x_2}{dt^2} + s_2 x_2 = \mu A p^2 e^{i p t}$$

which is an equation of forced vibration.

Thus the value of x_2 must be given as $x_2 = B e^{i p t}$ where B is another constant.

Substituting the values of $\frac{d^2 x_1}{dt^2}$, x_1 , $\frac{d^2 x_2}{dt^2}$ and x_2 in (6) and (7), we have

$$\left. \begin{aligned} -m_1 A p^2 - \mu B p^2 + s_1 A &= 0 \\ -m_2 B p^2 - \mu A p^2 + s_2 B &= 0 \end{aligned} \right\} \quad \dots \quad (8)$$

whence

$$\left. \begin{aligned} A(s_1 - m_1 p^2) &= \mu B p^2 \\ B(s_2 - m_2 p^2) &= \mu A p^2 \end{aligned} \right\} \quad \dots \quad (9)$$

$$\text{or, } \frac{s_1 - m_1 p^2}{\mu p^2} = \frac{\mu p^2}{s_2 - m_2 p^2}$$

$$\text{or, } \mu^2 p^4 = s_1 s_2 - s_1 m_2 p^2 - m_1 s_2 p^2 + m_1 m_2 p^4$$

$$\text{or, } \frac{\mu^2}{m_1 m_2} p^4 = \frac{s_1 s_2}{m_1 m_2} - \frac{s_1}{m_1} p^2 - \frac{s_2}{m_2} p^2 + p^4$$

Writing $\frac{\mu^2}{m_1 m_2} = k^2$ where k is known as the coefficient of coupling and $\frac{s_1}{m_1} = n_1^2$, $\frac{s_2}{m_2} = n_2^2$ where n_1 and n_2 are natural angular frequencies of each when coupling is absent, we get

$$p^4(1 - k^2) - p^2(n_1^2 + n_2^2) + n_1^2 n_2^2 = 0 \quad \dots \quad (10)$$

Solving for p^2 , we get

$$p^2 = \frac{(n_1^2 + n_2^2) \pm \sqrt{(n_1^2 + n_2^2)^2 - 4(1 - k^2)n_1^2 n_2^2}}{2(1 - k^2)}$$

Thus there are two possible frequencies of each constituting the coupled system.

Let the natural frequencies of the two constituting the coupled system be equal, then

$$p^2 = \frac{2n^2 \pm \sqrt{4n^4 k^2}}{2(1 - k^2)} \\ = \frac{n^2(1 \pm k)}{1 - k^2}$$

Thus the two angular frequencies are given by

$$p_1 = \frac{n}{\sqrt{1+k}} \quad \text{and} \quad p_2 = \frac{n}{\sqrt{1-k}}$$

Thus the two angular frequencies are respectively higher and lower than the natural angular frequency n of each; the difference in the two increases with the extent of coupling.

Now we can write as solution of (6) & (7)

$$\left. \begin{aligned} x_1 &= A_1 \sin(p_1 t + \alpha_1) + A_2 \sin(p_2 t + \alpha_2) \\ x_2 &= B_1 \sin(p_1 t + \beta_1) + B_2 \sin(p_2 t + \beta_2) \end{aligned} \right\} \dots (11)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants.

From (9) we have

$$\begin{aligned} B &= \frac{A}{\mu p^2} (s_1 - m_1 p^2) \\ &= \frac{m_1 A}{\mu p^2} \left(\frac{s_1}{m_1} - p^2 \right) \\ &= \frac{m_1 A}{\mu p^2} (n^2 - p^2) \end{aligned}$$

$$\text{If } p = p_1 = \frac{n}{\sqrt{1+k}}$$

$$\begin{aligned} B &= \frac{m_1 A}{\mu} \left(n^2 \frac{1+k}{n^2} - 1 \right) = \frac{k m_1 A}{\mu} \\ &= \frac{\mu}{\sqrt{m_1 m_2}} \cdot \frac{m_1}{\mu} A = A \sqrt{\frac{m_1}{m_2}} \end{aligned}$$

Again when $p = p_2 = \frac{n}{\sqrt{1-k}}$, by substitution we can get

$$B = -\sqrt{\frac{m_1}{m_2}} A.$$

Hence from (11), we get as solutions

$$\left. \begin{aligned} x_1 &= A_1 \sin(p_1 t + \alpha_1) + A_2 \sin(p_2 t + \alpha_2) \\ x_2 &= \sqrt{\frac{m_1}{m_2}} \left\{ A_1 \sin(p_1 t + \beta_1) - A_2 \sin(p_2 t + \beta_2) \right\} \end{aligned} \right\} \dots (12)$$

(A) Motion started by initial velocity: Let the motion be started by giving a velocity u_0 to m_1 at $t=0$. Thus at $t=0$.

$$\frac{dx_1}{dt} = u_0, \frac{dx_2}{dt} = 0, x_1 = x_2 = 0$$

From these initial conditions we obtain from (12)

$$\left. \begin{aligned} 0 &= A_1 \sin \alpha_1 + A_2 \sin \alpha_2 \\ u_0 &= p_1 A_1 \cos \alpha_1 + p_2 A_2 \cos \alpha_2 \\ 0 &= A_1 \sin \beta_1 - A_2 \sin \beta_2 \\ 0 &= p_1 A_1 \cos \beta_1 - p_2 A_2 \cos \beta_2 \end{aligned} \right\} \dots (13)$$

The above conditions will be satisfied if we take

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \text{ when } p_1 A_1 = p_2 A_2.$$

$$\text{Also } u_0 = 2p_1 A_1 = 2p_2 A_2$$

$$\text{whence } A_1 = \frac{u_0}{2p_1} = \frac{u_0 \sqrt{1+k}}{2n}$$

$$\text{and } A_2 = \frac{u_0}{2p_2} = \frac{u_0 \sqrt{1-k}}{2n}$$

Substituting the values of A_1, A_2 etc., we get from (12)

$$\left. \begin{aligned} x_1 &= \frac{u_0}{2n} \left(\sqrt{1+k} \sin \frac{nt}{\sqrt{1+k}} + \sqrt{1-k} \sin \frac{nt}{\sqrt{1-k}} \right) \\ x_2 &= \sqrt{\frac{m_1}{m_2}} \cdot \frac{u_0}{2n} \left(\sqrt{1+k} \sin \frac{nt}{\sqrt{1+k}} - \sqrt{1-k} \sin \frac{nt}{\sqrt{1-k}} \right) \end{aligned} \right\} (14)$$

When k is very small,

$$\frac{n}{\sqrt{1+k}} = \left(n - \frac{nk}{2} \right) \text{ and } \frac{n}{\sqrt{1-k}} = \left(n + \frac{nk}{2} \right)$$

$$\text{also } A_1 = A_2 = \frac{u_0}{2n}$$

Putting these values and simplifying, we obtain

$$\left. \begin{aligned} x_1 &= \frac{u_0}{n} \cos \frac{nk t}{2} \sin nt \\ x_2 &= -\sqrt{\frac{m_1}{m_2}} \cdot \frac{u_0}{n} \sin \frac{nk t}{2} \cos nt \end{aligned} \right\} \dots (15)$$

(B) Motion started by initial displacement:

Let m_1 be given an initial displacement x_0 at $t=0$. At that instant

$$x_2 = 0, \frac{dx_1}{dt} = 0, \frac{dx_2}{dt} = 0$$

Hence from (12), we have

$$\left. \begin{aligned} x_0 &= A_1 \sin \alpha_1 + A_2 \sin \alpha_2 \\ 0 &= p_1 A_1 \cos \alpha_1 + p_2 A_2 \cos \alpha_2 \\ 0 &= A_1 \sin \beta_1 - A_2 \sin \beta_2 \\ 0 &= p_1 A_1 \cos \beta_1 - p_2 A_2 \cos \beta_2 \end{aligned} \right\} \dots (16)$$

We see that conditions are satisfied if $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \pi/2$

when we have $A_1 = A_2 = \frac{x_0}{2}$.

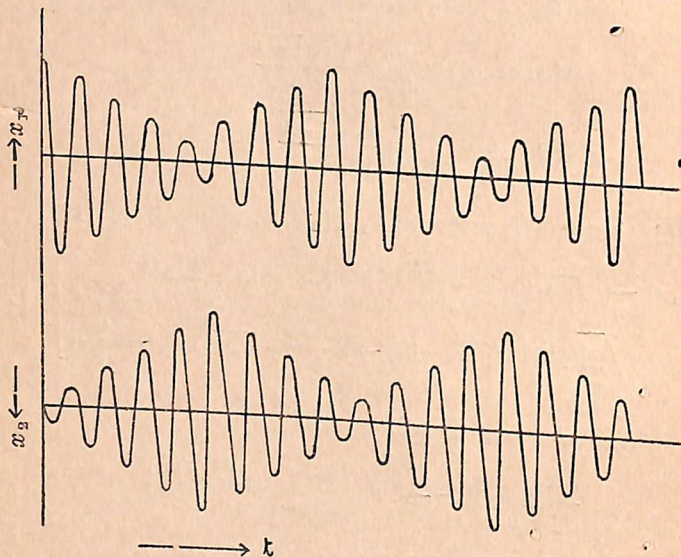


Fig. 15

Hence from (12) we get

$$x_1 = \frac{x_0}{2} \left(\cos \frac{nt}{\sqrt{1+k}} + \cos \frac{nt}{\sqrt{1-k}} \right)$$

$$\text{and } x_2 = \sqrt{\frac{m_1}{m_2}} \cdot \frac{x_0}{2} \left(\cos \frac{nt}{\sqrt{1+k}} - \cos \frac{nt}{\sqrt{1-k}} \right)$$

If k is very small

$$\left. \begin{aligned} x_1 &= x_0 \cos \frac{nkt}{2} \cos nt \\ x_2 &= x_0 \sqrt{\frac{m_1}{m_2}} \sin \frac{nkt}{2} \sin nt \end{aligned} \right\} \dots (17)$$

Equations (17) are illustrated by fig. 15.

(2) Vibrations : stiffness coupled :*

Let m_1 and m_2 be two point masses attached at points B and C of a string fixed at the ends A and D . Let the string be devoid of inertia and under a tension T which remains unchanged at small transverse displacements x_1 and x_2 of m_1 and m_2 in the plane of the diagram at any instant t . Let $AB = a$, $BC = b$ and $CD = c$, so that $l = a + b + c$.

From the figure, the force on m_1 in the positive direction

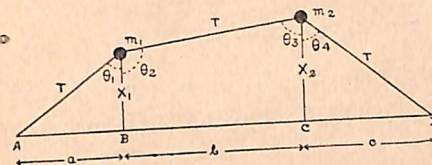


Fig. 16

of the displacement x_1 is $-T \cos \theta_1 - T \cos \theta_2 = -T \cdot \frac{x_1}{a} + T \cdot \frac{x_2 - x_1}{b}$, if x_1 and x_2 are small and neglecting variations in length.

Similarly force at the same instant on m_2 is $-T \cos \theta_3 - T \cos \theta_4 = -T \cdot \frac{x_2 - x_1}{b} - T \cdot \frac{x_2}{c}$.

Hence,

$$\left. \begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= -T \left(\frac{1}{a} + \frac{1}{b} \right) x_1 + \frac{T}{b} x_2 \\ \text{and } m_2 \frac{d^2 x_2}{dt^2} &= -T \left(\frac{1}{b} + \frac{1}{c} \right) x_2 + \frac{T}{b} x_1 \end{aligned} \right\} \dots (18)$$

We thus see that the motions of m_1 and m_2 are influenced by the displacements of m_2 and m_1 respectively and can rewrite equations (18) as

$$\left. \begin{aligned} m_1 \frac{d^2 x_1}{dt^2} + s_1 x_1 &= \mu x_2 \\ \text{and } m_2 \frac{d^2 x_2}{dt^2} + s_2 x_2 &= \mu x_1 \end{aligned} \right\} \dots (19)$$

*For problem of equal masses $m_1 = m_2$ and $a = b = c = l/3$, see page 46.

Such vibrations are stiffness coupled and can be written neglecting frictional forces as above.

That the coupling constant μ must be same in both the expressions of the coupled vibrations, can be shown by writing the energy equation which will be for two different values of coupling constants

$$\frac{1}{2}m_1 \left(\frac{dx_1}{dt}\right)^2 + \frac{1}{2}s_1 x_1^2 + \frac{\mu_1}{2}m_2 \left(\frac{dx_2}{dt}\right)^2 + \frac{1}{2}s_2 x_2^2 \}$$

$$-\mu_1 x_1 x_2 = \text{constant.}$$

Hence μ_2 must be equal to μ_1 or $\mu_1 = \mu_2 = \mu$ and the energy equation of stiffness coupled vibration is given by

$$\frac{1}{2}m_1 \left(\frac{dx_1}{dt}\right)^2 + \frac{1}{2}s_1 x_1^2 + \frac{1}{2}m_2 \left(\frac{dx_2}{dt}\right)^2 + \frac{1}{2}s_2 x_2^2$$

$$-\mu x_1 x_2 = \text{constant.}$$

Let us put $x_1 = Ae^{ipt}$ where A is a constant, then x_2 must be Be^{ipt} when B is another constant. Substituting the values of $\frac{d^2x_1}{dt^2}$ and $\frac{d^2x_2}{dt^2}$, x_1 and x_2 in (19) we get

$$\left. \begin{aligned} A(s_1 - m_1 p^2) &= \mu B \\ B(s_2 - m_2 p^2) &= \mu A \end{aligned} \right\} \dots \dots (20)$$

From which we get

$$m_1 m_2 p^4 - p^2 (s_2 m_1 + s_1 m_2) - \mu^2 + s_1 s_2 = 0$$

$$\text{or } p^4 - p^2 (n_1^2 + n_2^2) - k^2 + n_1^2 n_2^2 = 0$$

where $\frac{s_1}{m_1} = n_1^2$, $\frac{s_2}{m_2} = n_2^2$, $k = \frac{\mu}{\sqrt{m_1 m_2}}$, the coefficient of coupling.

From above we get

$$p^2 = \frac{(n_1^2 + n_2^2) \pm \sqrt{(n_1^2 + n_2^2)^2 + 4k^2 - 4n_1^2 n_2^2}}{2} \dots (21)$$

Thus p has two possible real positive values p_1 and p_2 , one higher than the other. Hence solutions for displacements will be

$$x_1 = A_1 e^{ip_1 t} + A_2 e^{ip_2 t}$$

$$x_2 = B_1 e^{ip_1 t} + B_2 e^{ip_2 t}$$

Let m_1 and m_2 have the same natural period, i. e.,

$$\frac{m_1}{s_1} = \frac{m_2}{s_2} \quad \text{or } n_1^2 = n_2^2.$$

We have from (21) putting $n_1 = n_2 = n$

$$p_1^2 = n^2 + k \quad \text{and } p_2^2 = n^2 - k$$

If k is small, $p_1 = n + \frac{k}{2n}$ and $p_2 = n - \frac{k}{2n}$.

From (20) we get

$$B = \frac{A}{\mu} (s_1 - m_1 p^2) = \frac{A m_1}{\mu} (n_1^2 - p^2)$$

$$= \frac{A m_1}{k \sqrt{m_1 m_2}} (n_1^2 - p^2)$$

$$= A \cdot \sqrt{\frac{m_1}{m_2}} (n_1^2 - p^2) / k$$

If $n_1 = n_2 = n$, then for $p_1^2 = n^2 + k$, we get

$$B = -\sqrt{\frac{m_1}{m_2}} \cdot A.$$

For $p_2^2 = n^2 - k$, we have similarly

$$B = \sqrt{\frac{m_1}{m_2}} \cdot A$$

Hence we can write for such a system

$$\left. \begin{aligned} x_1 &= a_1 \sin(p_1 t + \alpha_1) + a_2 \sin(p_2 t + \alpha_2) \\ x_2 &= \sqrt{\frac{m_1}{m_2}} \left\{ -a_1 \sin(p_1 t + \beta_1) + a_2 \sin(p_2 t + \beta_2) \right\} \end{aligned} \right\} \dots (22)$$

Let us suppose the motion is started by an initial displacement x_0 of m_1 at $t=0$, when $x_2=0$,

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = 0.$$

From these initial conditions we get

$$x_0 = a_1 \sin \alpha_1 + a_2 \sin \alpha_2$$

$$0 = p_1 a_1 \cos \alpha_1 + p_2 a_2 \cos \alpha_2$$

$$0 = -a_1 \sin \beta_1 + a_2 \sin \beta_2$$

$$0 = -p_1 a_1 \cos \beta_1 + p_2 a_2 \cos \beta_2$$

The above will be satisfied if we take $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \pi/2$

when we get $a_1 = a_2 = \frac{x_0}{2}$.

Hence from (22), we get

$$\begin{aligned} x_1 &= \frac{x_0}{2} (\cos p_1 t + \cos p_2 t) \\ &= \frac{x_0}{2} \left\{ \cos \left(n + \frac{k}{2n} \right) t + \cos \left(n - \frac{k}{2n} \right) t \right\} \\ &= x_0 \cos \frac{k}{2n} t \cos nt \end{aligned}$$

$$\begin{aligned} \text{and } x_2 &= \frac{x_0}{2} \sqrt{\frac{m_1}{m_2}} \left\{ -\cos \left(n + \frac{k}{2n} \right) t + \cos \left(n - \frac{k}{2n} \right) t \right\} \\ &= x_0 \sqrt{\frac{m_1}{m_2}} \sin \frac{k}{2n} t \sin nt \end{aligned}$$

Expressions for x_1 and x_2 are illustrated by fig. (15) when m_1 and m_2 are nearly equal.

3. Vibration of two equal masses attached to a uniform string under tension :

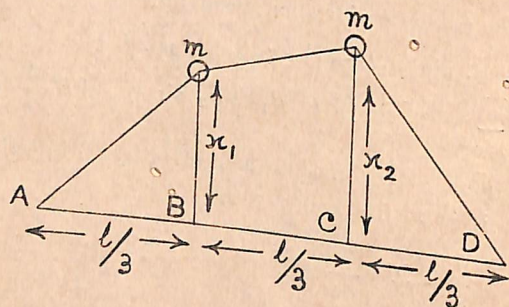


Fig. 17

Let the string of negligible mass be divided into 3 equal parts, AB, BC & CD. Let equal masses m be attached to B & C. Let x_1 and x_2 be the displacements of the masses at any instant t where x_1 and x_2 are small.

Then considering that the tension T remains unchanged

$$\begin{aligned} m \frac{d^2 x_1}{dt^2} &= -\frac{T x_1}{l/3} + T \frac{x_2 - x_1}{l/3} \\ &= -\frac{3T x_1}{l} + 3T \frac{x_2 - x_1}{l} \\ &= -6T \frac{x_1}{l} + 3T \frac{x_2}{l} \\ m \frac{d^2 x_2}{dt^2} &= -T \frac{x_2 - x_1}{l/3} - T \frac{x_2}{l/3} \\ &= -6T \frac{x_2}{l} + 3T \frac{x_1}{l} \end{aligned}$$

From the above

$$\begin{aligned} m \frac{d^2}{dt^2} (x_1 + x_2) &= -\frac{3T}{l} (x_1 + x_2) \\ m \frac{d^2}{dt^2} (x_1 - x_2) &= -\frac{9T}{l} (x_1 - x_2) \end{aligned}$$

Re-arranging

$$\begin{aligned} \frac{d^2}{dt^2} (x_1 + x_2) + \frac{3T}{lm} (x_1 + x_2) &= 0 \\ \frac{d^2}{dt^2} (x_1 - x_2) + \frac{9T}{lm} (x_1 - x_2) &= 0 \end{aligned}$$

whence

$$\begin{aligned} x_1 + x_2 &= a_1 \cos \sqrt{\frac{3T}{lm}} t + b_1 \sin \sqrt{\frac{3T}{lm}} t \\ x_1 - x_2 &= a_2 \cos \sqrt{\frac{9T}{lm}} t + b_2 \sin \sqrt{\frac{9T}{lm}} t \\ \therefore x_1 &= \frac{1}{2} \left(a_1 \cos \sqrt{\frac{3T}{lm}} t + b_1 \sin \sqrt{\frac{3T}{lm}} t + a_2 \cos \sqrt{\frac{9T}{lm}} t \right. \\ &\quad \left. + b_2 \sin \sqrt{\frac{9T}{lm}} t \right) \\ \text{and } x_2 &= \frac{1}{2} \left(a_1 \cos \sqrt{\frac{3T}{lm}} t + b_1 \sin \sqrt{\frac{3T}{lm}} t - a_2 \cos \sqrt{\frac{9T}{lm}} t \right. \\ &\quad \left. - b_2 \sin \sqrt{\frac{9T}{lm}} t \right) \end{aligned}$$

Thus both the masses may vibrate with angular frequencies

$$\sqrt{\frac{3T}{lm}} \quad \& \quad \sqrt{\frac{9T}{lm}}$$

4. Forced vibration of a stiffness controlled coupled system :

If frictional forces are taken into account the equations of motion of two systems of masses m_1 and m_2 can be written as

$$m_1 \frac{d^2 x_1}{dt^2} + k_1 \frac{dx_1}{dt} + s_1 x_1 - \mu x_2 = F e^{i p t} \quad \dots (23)$$

$$m_2 \frac{d^2 x_2}{dt^2} + k_2 \frac{dx_2}{dt} + s_2 x_2 - \mu x_1 = 0 \quad \dots (24)$$

where k_1, k_2 are damping constants, s_1, s_2 the stiffness constants and μ the coefficient of coupling and the periodic force is $F e^{i p t}$ acting on the 1st system coupled to second system. We must have solutions for the displacements in the steady state as

$$x_1 = A e^{i p t} \text{ and } x_2 = B e^{i p t}$$

Then substituting in (23) & (24) we have

$$A(s_1 + k_1 i p - m_1 p^2) - \mu B = F \quad \dots (25)$$

$$B(s_2 + k_2 i p - m_2 p^2) = \mu A \quad \dots (26)$$

From above we have

$$B \left(\frac{s_2}{m_2} + \frac{k_2 i p}{m_2} - p^2 \right) \left(\frac{s_1}{m_1} + \frac{k_1 i p}{m_1} - p^2 \right) - \frac{\mu^2 B}{m_1 m_2} = \frac{\mu F}{m_1 m_2}$$

$$\text{or, } B(n_2^2 - p^2 + 2b_1 i p)(n_1^2 - p^2 + 2b_2 i p) - k^2 B = \frac{k}{\sqrt{m_1 m_2}} F \quad \dots (27)$$

$$\text{where } \frac{s_1}{m_1} = n_1^2, \frac{s_2}{m_2} = n_2^2, \frac{k_1}{m_1} = 2b_1, \frac{k_2}{m_2} = 2b_2$$

$$\text{and coupling coefficient } k = \frac{\mu}{\sqrt{m_1 m_2}}$$

From (27) after simplification we obtain

$$B = \frac{\frac{k}{\sqrt{m_1 m_2}} F}{(n_2^2 - p^2)(n_1^2 - p^2) - (4b_1 b_2 p^2 + k^2) + 2ip\{b_1(n_1^2 - p^2) + b_2(n_2^2 - p^2)\}}$$

$$\text{We can write } B = \frac{C}{X + iY}$$

$$= \frac{C(X - iY)}{X^2 + Y^2} = \frac{C}{\sqrt{X^2 + Y^2}} \cdot e^{-i\theta}$$

$$\text{where } \tan \theta = \frac{Y}{X}$$

$$\text{Thus } B = \frac{\frac{k}{\sqrt{m_1 m_2}} F e^{-i\theta}}{\sqrt{\{(n_2^2 - p^2)(n_1^2 - p^2) - (4b_1 b_2 p^2 + k^2)\}^2 + 4p^2\{b_1(n_1^2 - p^2) + b_2(n_2^2 - p^2)\}^2}} \quad \dots (28)$$

$$\text{and } \tan \theta = \frac{2p\{b_1(n_1^2 - p^2) + b_2(n_2^2 - p^2)\}}{(n_1^2 - p^2)(n_2^2 - p^2) - (4b_1 b_2 p^2 + k^2)} \quad \dots (29)$$

From (28) & (29), we obtain the amplitude of vibration of m_2 and we see that the vibration lags behind the periodic force by an angle θ given by (29).

Also from (26), we have

$$\begin{aligned} A &= \frac{B}{\mu} (s_2 + k_2 i p - m_2 p^2) = \frac{m_2 B}{\mu} (n_2^2 - p^2 + 2b_2 i p) \\ &= \frac{m_2 B}{\mu} \sqrt{(n_2^2 - p^2)^2 + 4b_2^2 p^2} e^{i\theta^1} \\ &= \frac{B}{k} \sqrt{\frac{m_2}{m_1}} \cdot \sqrt{(n_2^2 - p^2)^2 + 4b_2^2 p^2} e^{i\theta^1} \quad \dots (30) \end{aligned}$$

$$\text{where } \tan \theta^1 = \frac{2b_2 p}{n_2^2 - p^2} \quad \dots (31)$$

Expressions (30) & (31) give the value of A which can be found out substituting the value of B . We see that the vibration of m_1 is ahead of that of m_2 by an angle $\theta^1 = \tan^{-1} \frac{2b_2 p}{n_2^2 - p^2}$

In the special case when $n_1 = n_2 = n$, $b_1 = b_2 = 0$ and $\therefore \theta = \theta^1 = 0$, the amplitude of steady vibrations will be given by

$$B = \frac{\frac{k}{\sqrt{m_1 m_2}} F}{(n^2 - p^2)^2 - k^2}$$

The amplitude B becomes infinite at $(n^2 - p^2)^2 = k^2$

$$\text{i.e., at } p = n \pm \frac{k}{2n} \quad \left(\text{for small values of } \frac{k}{n^2} \right).$$

$$\text{Since } A = \frac{B}{k} \sqrt{\frac{m_2}{m_1}} (n^2 - p^2)$$

this also becomes infinite at the above two values of p .

In actual cases b_1 and b_2 will never be zero and, therefore, A and B will never be infinite. A & B can be plotted with p and curve for each will show two peaks at certain values of p determined by n_1, n_2, b_1, b_2 and k .

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$$B \left(\frac{s_2}{m_2} + \frac{k_2 i p}{m_2} - p^2 \right) \left(\frac{s_1}{m_1} + \frac{k_1 i p}{m_1} - p^2 \right) - \frac{\mu^2 B}{m_1 m_2} = \frac{\mu F}{m_1 m_2}$$

$$\text{or, } B(n_2^2 - p^2 + 2b_1 i p)(n_1^2 - p^2 + 2b_2 i p) - k^2 B = \frac{k}{\sqrt{m_1 m_2}} \cdot F \quad \dots (27)$$

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From (27) after simplification we obtain

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$$\text{and } \tan \theta = \frac{2p\{b_1(n_1^2 - p^2) + b_2(n_2^2 - p^2)\}}{(n_1^2 - p^2)(n_2^2 - p^2) - (4b_1 b_2 p^2 + k^2)} \quad \dots (29)$$

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$$= \frac{m_2 B}{\mu} \sqrt{(n_2^2 - p^2)^2 + 4b_2^2 p^2} e^{i\theta^1}$$

$$= \frac{B}{k} \sqrt{\frac{m_2}{m_1}} \cdot \sqrt{(n_2^2 - p^2)^2 + 4b_2^2 p^2} e^{i\theta^1} \quad \dots (30)$$

$$\text{where } \tan \theta^1 = \frac{2b_2 p}{n_2^2 - p^2} \quad \dots (31)$$

Expressions (30) & (31) give the value of A which can be found out substituting the value of B . We see that the vibration of m_1 is ahead of that of m_2 by an angle $\theta^1 = \tan^{-1} \frac{2b_2 p}{n_2^2 - p^2}$

In the special case when $n_1 = n_2 = n$, $b_1 = b_2 = 0$ and $\therefore \theta = \theta^1 = 0$, the amplitude of steady vibrations will be given by

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The amplitude B becomes infinite at $(n^2 - p^2)^2 = k^2$

$$\text{i.e., at } p = n \pm \frac{k}{2n} \quad \left(\text{for small values of } \frac{k}{n^2} \right).$$

$$\text{Since } A = \frac{B}{k} \sqrt{\frac{m_2}{m_1}} (n^2 - p^2)$$

this also becomes infinite at the above two values of p .

In actual cases b_1 and b_2 will never be zero and, therefore, A and B will never be infinite. A & B can be plotted with p and curve for each will show two peaks at certain values of p determined by n_1, n_2, b_1, b_2 and k .

CHAPTER V

FOURIER'S THEOREM

1. Fourier's Theorem : When several simple harmonic vibrations of commensurate periods combine, they may produce some type of periodic vibration ; again any complex vibration may be analysed into simple harmonic vibrations of commensurate periods. **Fourier's Theorem** is of great importance in the synthesis as well as analysis of periodic motions. When applied to the problems in sound, it can be stated in the following way.

"Any finite periodic motion can be expressed as the sum of a series of simple harmonic motions of commensurate periods."

Mathematically it can be stated that if y is any periodic function of time t , then

$$y = a_0 + a_1 \cos nt + a_2 \cos 2nt + \dots + a_s \cos snt + \dots \\ + b_1 \sin nt + b_2 \sin 2nt + \dots + b_s \sin snt + \dots \quad (1)$$

The above series known as the Fourier series can also be written as a series of only sine terms or only cosine terms. Thus if $a_s = A_s \cos \alpha_s$ and $b_s = -A_s \sin \alpha_s$.

$$A_s = \sqrt{a_s^2 + b_s^2} \quad \text{and} \quad \tan \alpha_s = -\frac{b_s}{a_s} \dots \quad (2)$$

So we may write (1) as

$$y = a_0 + A_1 \cos (nt + \alpha_1) + A_2 \cos (2nt + \alpha_2) + \dots \\ + A_s \cos (snt + \alpha_s) + \dots \quad (3)$$

The s th harmonic vibration is represented by $A_s \cos (snt + \alpha_s)$ the fundamental one being $A_1 \cos (nt + \alpha_1)$. Thus if $a_1, a_2, \dots, b_1, b_2, \dots$ etc., in series (1) are found out, we can calculate with the help of (2) the amplitudes and phase angles of the fundamental and the different overtones represented by the series (3).

3. Evaluation of the coefficients : In order to find out a_0 in (1) let both sides be integrated with respect to t from 0 to T , T being the periodic time. From (1) it is clear that the displacement y will repeat a cycle when nt changes by 2π ; hence periodic time $T = \frac{2\pi}{n}$.

$$\text{Then } \int_0^T y dt = \int_0^T a_0 dt + \int_0^T \sum_{s=1}^{s=\infty} a_s \cos snt dt \\ + \int_0^T \sum_{s=1}^{s=\infty} b_s \sin snt dt.$$

$$\text{But } \int_0^T \cos snt dt = \frac{1}{sn} [\sin snt]_0^T = \frac{1}{sn} [\sin 2s\pi - \sin 0] = 0.$$

$$\text{Also } \int_0^T \sin snt dt = -\frac{1}{sn} [\cos snt]_0^T \\ = -\frac{1}{sn} (\cos 2s\pi - \cos 0) = 0$$

$$\therefore a_0 T = \int_0^T y dt$$

$$\text{or, } a_0 = \frac{1}{T} \int_0^T y dt \quad \dots \quad (4)$$

To find out a_s , let both sides of (1) be multiplied by $\cos snt$ and integrated with respect to t from $t=0$ to $t=T$.

Then

$$\begin{aligned} & \int_0^T y \cos snt \, dt \\ &= \int_0^T (a_0 + a_1 \cos nt + a_2 \cos 2nt + \dots + a_s \cos snt + \dots) \cos snt \, dt \\ &+ \int_0^T (b_1 \sin nt + b_2 \sin 2nt + \dots + b_s \sin snt + \dots) \cos snt \, dt \dots (5) \end{aligned}$$

Now $\int_0^T \cos knt \cos snt \, dt$, k being an integer

$$= \frac{1}{2} \left[\int_0^T \cos (k+s) nt \, dt + \int_0^T \cos (k-s) nt \, dt \right] \dots (6)$$

The 1st integral is zero for all integral values of k . The second integral is also zero for all integral values of k , except $k=s$ when it is equal to T . Again

$$\int_0^T \sin knt \cos snt \, dt = \frac{1}{2} \left[\int_0^T \sin (k+s) nt \, dt + \int_0^T \sin (k-s) nt \, dt \right]$$

$= 0$ for all integral values of k including $k=s$.

$$\text{Thus } \int_0^T (a_0 + a_1 \cos nt + a_2 \cos 2nt + \dots + a_s \cos snt + \dots) \cos snt \, dt$$

$$+ \int_0^T (b_1 \sin nt + b_2 \sin 2nt + \dots + b_s \sin snt) \cos snt \, dt$$

$$= a_s \frac{T}{2}$$

$$\therefore a_s = \frac{2}{T} \int_0^T y \cos snt \, dt \dots (7)$$

In the same way multiplying both sides of (1) by $\sin snt$ and then integrating with respect to t from 0 to T , we shall get

$$b_s = \frac{2}{T} \int_0^T y \sin snt \, dt \dots (8)$$

In simple cases, $y=f(t)$ is known and hence the coefficients can be calculated by performing the integrations (4), (7) and (8). But in most cases, the function is not known and then we may suppose the complex periodic curve to consist of small portions of straight lines within known limits of time; any of the above integrals is then the sum of the integrals over the different component straight portions. The process is, however, very tedious and there are easier approximate methods. Moreover, the integrations can be carried out by appliances known as harmonic analysers.

3. Analysis by Fourier's theorem :

(a) Let the displacement curve of a vibrating particle be given by $y=0$ from $t=0$ to $t=T/2$ and $y=k$ from $t=T/2$ to $t=T$.

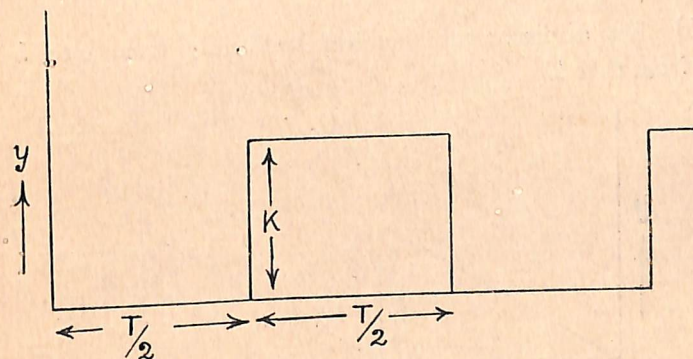


Fig. 18

$$\text{Then } a_0 = \frac{1}{T} \int_0^T y \, dt = \int_0^{T/2} y \, dt + \frac{1}{T} \int_{T/2}^T y \, dt = \frac{1}{T} \left(\int_{T/2}^T k \, dt \right) = \frac{k}{2}$$

$$\frac{a_s}{2} = \frac{1}{T} \int_0^T y \cos snt \, dt = \frac{k}{T} \int_{T/2}^T \cos snt \, dt$$

$$= \frac{k}{snT} \left[\sin snt \right]_{T/2}^T$$

$$= \frac{k}{snT} [\sin 2s\pi - \sin s\pi] = 0.$$

$$\frac{b_s}{2} = \frac{1}{T} \int_0^T y \sin snt \, dt = \frac{k}{T} \int_{T/2}^T \sin snt \, dt$$

$$= -\frac{k}{snT} \left[\cos snt \right]_{T/2}^T$$

$$= -\frac{k}{snT} [\cos 2s\pi - \cos s\pi]$$

If s is even, $b_s = 0$

$$\text{and if } s \text{ is odd } \frac{b_s}{2} = -\frac{2k}{snT} = -\frac{k}{s\pi}$$

$$\text{Hence, } y = \frac{k}{2} - \frac{k}{\pi} \left[\sin nt + \frac{1}{3} \sin 3nt + \frac{1}{5} \sin 5nt + \dots \right]$$

(b) Let displacement increase uniformly from 0 to k in time from 0 to T .

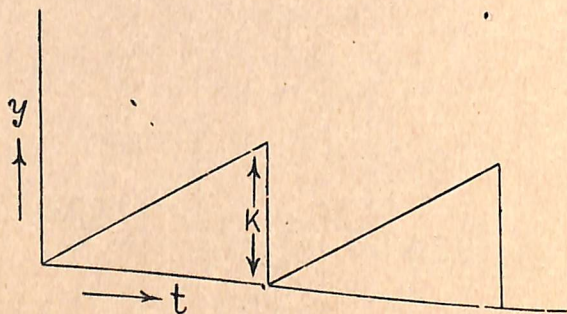


Fig. 19

$$\text{Here } \frac{y}{t} = \frac{k}{T} \therefore y = \frac{k}{T} t$$

$$\text{Hence } a_0 = \frac{1}{T} \int_0^T y \, dt$$

$$= \frac{k}{T^2} \left[\frac{t^2}{2} \right]_0^T = \frac{k}{2}$$

$$a_s = \frac{2}{T} \int_0^T y \cos snt \, dt = \frac{2k}{T^2} \int_0^T t \cos snt \, dt$$

$$= \frac{2k}{T^2} \left[\frac{t}{sn} (\sin snt) \right]_0^T - \frac{2k}{T^2 sn} \int_0^T \sin snt \, dt \quad (\text{Integrating by parts}).$$

$$= \frac{2k}{T^2} \left[\frac{t}{sn} \sin snt + \frac{1}{s^2 n^2} \cos snt \right]_0^T$$

$$= 0$$

$$b_s = \frac{2}{T} \int_0^T y \sin snt \, dt$$

$$= \frac{2k}{T^2} \left[-\frac{T}{sn} \cos 2\pi s + \frac{1}{s^2 n^2} \sin 2\pi s \right]$$

$$= -\frac{2k}{snT}$$

$$= -\frac{k}{s\pi} \quad \left(\because n = \frac{2\pi}{T} \right)$$

$$\text{Thus } y = \frac{k}{2} - \frac{k}{\pi} \left(\sin nt + \frac{1}{3} \sin 3nt + \frac{1}{5} \sin 5nt + \dots \right)$$

$$+ \dots \frac{1}{s} \sin snt + \dots$$

Hence all the harmonics are present.

(c) The displacement decreases uniformly from k to zero in time from $t=0$ to $t=T/2$ and then increases uniformly from zero to k in time from $T/2$ to T

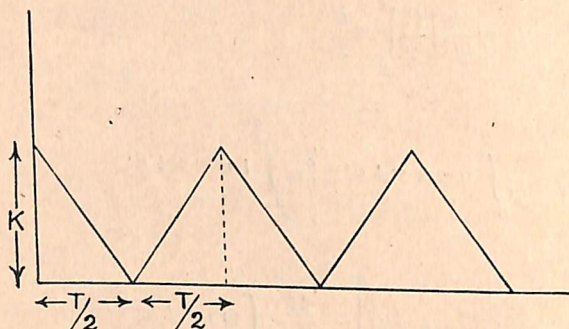


Fig. 20

When $0 < t < T/2$

$$\frac{y}{T/2 - t} = \frac{k}{T/2} \text{ or } y = k - \frac{2k}{T} t$$

when $T/2 < t < T$

$$\frac{y}{t - T/2} = \frac{k}{T/2} \text{ or } y = \frac{2kt}{T} - k$$

$$\text{Hence } a_0 = \frac{1}{T} \int_0^T y dt$$

$$\begin{aligned} &= \int_0^{T/2} \left(k - \frac{2k}{T} t \right) dt + \frac{1}{T} \int_{T/2}^T \left(\frac{2k}{T} t - k \right) dt \\ &= \frac{k}{2} \end{aligned}$$

$$a_s = \frac{2}{T} \int_0^{T/2} \left(k - \frac{2k}{T} t \right) \cos snt dt + \frac{2}{T} \int_{T/2}^T \left(\frac{2k}{T} t - k \right) \cos snt dt$$

$$= \frac{2}{T} \left[\left(\frac{k}{sn} \sin snt \right)_{T/2} - \left(\frac{k}{sn} \sin snt \right)_{T/2} \right. \\ \left. - \frac{7k}{T} \int_0^{T/2} t \cos snt \, dt + \frac{2k}{T} \int_{T/2}^T t \cos snt \, dt \right]$$

The 1st two integrals are zero.

$$\therefore a_s = -\frac{4k}{T^2} \int_0^{T/2} t \cos snt \, dt + \frac{4k}{T^2} \int_{T/2}^T t \cos snt \, dt$$

$$\text{But } \int t \cos snt \, dt$$

$$= \frac{t}{sn} \sin snt + \frac{1}{s^2 n^2} \cos snt + \text{constant.}$$

$$\therefore a_s = -\frac{4k}{T^2} \left(\frac{t}{sn} \sin snt + \frac{1}{s^2 n^2} \cos snt \right)_{T/2} + \frac{4k}{T^2} \left(\frac{t}{sn} \sin snt \right. \\ \left. + \frac{1}{s^2 n^2} \cos snt \right)_{T/2}$$

$$= -\frac{4k}{T^2} \left(\frac{1}{s^2 n^2} \cos s\pi - \frac{1}{s^2 n^2} \right) + \frac{4k}{T^2} \left(\frac{1}{s^2 n^2} - \frac{1}{s^2 n^2} \cos s\pi \right)$$

$$= -\frac{8k}{T^2 s^2 n^2} (\cos s\pi - 1)$$

Thus when s is even, a_s is zero and when s is odd

$$a_s = \frac{16k}{T^2 s^2 n^2}$$

$$\text{Now } b_s = \frac{2}{T} \left[\int_0^{T/2} \left(k - \frac{2k}{T} t \right) \sin snt \, dt + \int_{T/2}^T \left(\frac{2k}{T} t - k \right) \sin snt \, dt \right]$$

$$= \frac{2k}{T} \left[-\frac{k}{sn} \cos snt \right]_0^{T/2} + \left[\frac{k}{sn} \cos snt \right]_{T/2}^T$$

$$- \frac{2k}{T} \left[\int_0^{T/2} t \sin snt \, dt - \int_{T/2}^T t \sin snt \, dt \right]$$

$$= \frac{2k}{T} \left[-\frac{k}{sn} (\cos s\pi - 1) + \frac{k}{sn} (1 - \cos s\pi) \right]$$

$$- \frac{2k}{T} \left[\int_0^{T/2} t \sin snt \, dt - \int_{T/2}^T t \sin snt \, dt \right]$$

$$\text{Again } \int_0^{T/2} t \sin snt \, dt - \int_{T/2}^T t \sin snt \, dt$$

$$= \left(-\frac{t}{sn} \cos snt + \frac{1}{s^2 n^2} \sin snt \right)_0^{T/2} - \left(-\frac{t}{sn} \cos snt + \frac{1}{s^2 n^2} \sin snt \right)_{T/2}^T$$

$$= \left(-\frac{T}{2sn} \cos s\pi \right) - \left(-\frac{T}{sn} + \frac{T}{2sn} \cos s\pi \right)$$

$$= \frac{T}{sn} - \frac{T}{sn} \cos s\pi$$

$$\therefore b_s = \frac{2}{T} \left[-\frac{2k}{sn} \cos s\pi + \frac{2k}{sn} - \frac{2k}{T} \left(\frac{T}{sn} - \frac{T}{sn} \cos s\pi \right) \right]$$

$$= \frac{2}{T} \left[-\frac{2k}{sn} \cos s\pi + \frac{2k}{sn} - \frac{2k}{sn} + \frac{2k}{sn} \cos s\pi \right]$$

$$= 0.$$

$$\text{Now as } a_s = \frac{16k}{T^2 s^2 n^2} = \frac{4k}{s^2 \pi^2}$$

$$y = \frac{k}{2} + \frac{4k}{\pi^2} \left[\frac{1}{1^2} \cos nt + \frac{1}{3^2} \cos 3nt + \dots \right. \\ \left. + \frac{1}{s^2} \cos snt + \dots \right]$$

(d) $y = A \sin nt$ from $t=0$ to $t=T/2$; $y=0$ from $t=T/2$ to $t=T$:

(Approximate curve of this nature is obtained in half-wave rectification of alternating current)

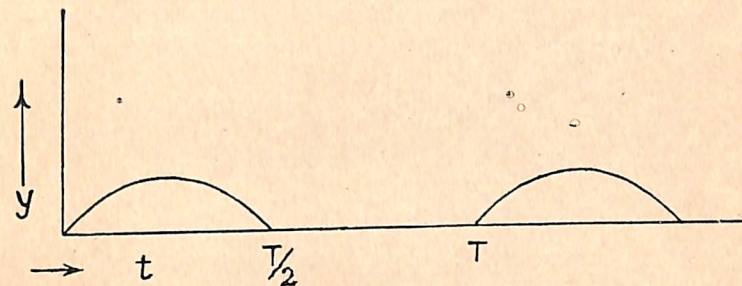


Fig. 21

$$a_0 = \frac{A}{T} \int_0^T \sin nt \, dt = \frac{A}{T} \int_0^{T/2} \sin nt \, dt$$

$$= -\frac{A}{Tn} \left[\cos nt \right]_0^{T/2}$$

$$= -\frac{A}{2\pi} \left[\cos \pi - 1 \right]$$

$$= \frac{A}{\pi}$$

$$a_s = \frac{2}{T} \int_0^{T/2} A \sin nt \cos snt \, dt$$

$$= -\frac{A}{T(1+s)n} \left[\cos (s+1)\pi - 1 \right]$$

$$- \frac{A}{T(1-s)n} \left[\cos (1-s)\pi - 1 \right]$$

$$= -\frac{A}{2\pi(1+s)} \left[\cos (1+s)\pi - 1 \right]$$

$$+ \frac{A}{2\pi(s-1)} \left[\cos (s-1)\pi - 1 \right]$$

Putting $s=1$, the first term in the expression is zero.

$$\text{since } \frac{\cos(s-1)\pi - 1}{2\pi(s-1)} = \frac{-2 \sin^2 \frac{(s-1)\pi}{2}}{2\pi(s-1)}$$

$$= 0 \text{ when } s \rightarrow 1$$

$$\text{Thus } a_1 = 0$$

$$a_2 = A \left[\frac{1}{3\pi} - \frac{1}{\pi} \right]$$

$$a_3 = 0$$

$$a_4 = A \left[\frac{1}{5\pi} - \frac{1}{3\pi} \right]$$

$$a_5 = 0$$

$$a_6 = A \left[\frac{1}{7\pi} - \frac{1}{5\pi} \right]$$

$$\& a_s = A \left[\frac{1}{(s+1)\pi} - \frac{1}{(s-1)\pi} \right]$$

where s is an even integer.

$$\text{Now, } b_s = \frac{A}{T} \int_0^{T/2} \sin nt \sin snt \, dt$$

$$= \frac{A}{Tn(s-1)} \left[\sin(s-1)nt \right]_0^{T/2} - \frac{A}{Tn(s+1)} \left[\sin(s+1)nt \right]_0^{T/2}$$

$$\text{Hence } b_s = 0 \text{ except } b_1 \text{ which is } \frac{A}{2}$$

$$\text{Thus } y = \frac{A}{\pi} + \frac{A}{\pi} \left\{ \left(\frac{1}{3} - \frac{1}{1} \right) \cos 2nt + \left(\frac{1}{5} - \frac{1}{3} \right) \cos 4nt \right. \\ \left. + \left(\frac{1}{7} - \frac{1}{5} \right) \cos 6nt + \dots \right\} + \frac{A}{2} \sin nt$$

4. Graphical method of analysis :

Suppose we have time-displacement curve for a vibration of a complex type and the periodic time is known. We know that displacement is given by

$$y = a_0 + \sum_{s=1}^{\infty} a_s \cos snt + \sum_{s=1}^{\infty} b_s \sin snt$$

Putting $s=1$, the first term in the expression is zero.

$$\text{since } \frac{\cos(s-1)\pi - 1}{2\pi(s-1)} = \frac{-2 \sin^2 \frac{(s-1)\pi}{2}}{2\pi(s-1)} \\ = 0 \text{ when } s \rightarrow 1$$

Thus $a_1 = 0$

$$a_2 = A \left[\frac{1}{3\pi} - \frac{1}{\pi} \right]$$

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$$a_6 = A \left[\frac{1}{7\pi} - \frac{1}{5\pi} \right]$$

$$\& a_s = A \left[\frac{1}{(s+1)\pi} - \frac{1}{(s-1)\pi} \right]$$

where s is an even integer.

$$\text{Now, } b_s = \frac{2A}{T} \int_0^{T/2} \sin nt \sin snt \, dt$$

$$= \frac{A}{Tn(s-1)} \left[\sin(s-1)nt \right]_0^{T/2} - \frac{A}{Tn(s+1)} \left[\sin(s+1)nt \right]_0^{T/2}$$

Hence $b_s = 0$ except b_1 which is $\frac{A}{2}$

$$\text{Thus } y = \frac{A}{\pi} + \frac{A}{\pi} \left(\frac{1}{3} - \frac{1}{1} \right) \cos 2nt + \left(\frac{1}{5} - \frac{1}{3} \right) \cos 4nt \\ + \left(\frac{1}{7} - \frac{1}{5} \right) \cos 6nt + \dots + \frac{A}{2} \sin nt$$

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Suppose we have time-displacement curve for a vibration of a complex type and the periodic time is known. We know that displacement is given by

$$y = a_0 + \sum_{s=1}^{\infty} a_s \cos snt + \sum_{s=1}^{\infty} b_s \sin snt$$

$$\text{where } a_0 = \frac{1}{T} \int_0^T y \, dt, \quad a_s = \frac{2}{T} \int_0^T y \cos snt \, dt \text{ and}$$

$$b_s = \frac{2}{T} \int_0^T y \sin snt \, dt.$$

To know a_0 , the area between the curve and x axis from $t=0$, to $t=T$ is found out by a planimeter ; then a_0 is the area calculated divided by the periodic time T .

To find a_s , y is multiplied by $\cos snt$ and the product is plotted against t . The area bounded by the curve and x -axis from $t=0$, to $t=T$ will be $\int_0^T y \cos snt \, dt$ from which a_s is found

out. In a similar way the product of $\sin snt$ and the displacement y for different values of t is plotted against t and the area between the curve and x axis from $t=0$ to $t=T$ will give the value of $\int_0^T y \sin snt \, dt$ and hence b_s can be calculated. The method is however very laborious.

5. Importance of Fourier's theorem in Sound : When a source of sound vibrates simple harmonically, the sound emitted by it is said to be pure. But in most cases, periodic vibrations are complex. Two sound sources vibrating with the same period will be no doubt concordant, but the quality of each will be determined by the component simple harmonic vibrations of which they may be supposed to be composed. If the displacement curve of a periodically vibrating body, say, any point on the string of a musical instrument is known, we can find out the different overtones present and calculate their amplitudes, phases etc. In other words, the theorem enables us to study quantitatively the quality of a musical sound.

CHAPTER VI

 LONGITUDINAL WAVES IN AN
ELASTIC MEDIUM

1. **Progressive waves :** Let A be the outer surface of a prong of a tuning fork in an elastic medium like air. Let us consider the state of particles of the medium in front of A when the fork vibrates. First, let us consider that the prong be given a sudden velocity towards the right. The layer of air in front of A undergoes a compression which will react on a contiguous layer producing there a compression, which in turn will act on the next layer and so on. Thus a pulse of compression will travel towards the right. Again if we consider the fork to be suddenly moving towards the left with a large velocity, there will be a fall of pressure in the layer of air in front of A . Particles from next layer to the right will move towards the left due to this fall of pressure and this process will go on from layer to layer, and thus a pulse of rarefaction will move rightwards. The velocity of propagation of this compression and rarefaction will depend on the elastic properties and density of the medium.

If, however, the vibrations of the fork are simple harmonic, a particle of the medium at any point in front of the fork will have simple harmonic motion about its mean position (its position when undisturbed) along the direction of propagation of the wave. Since the vibration of the particle is forced by that of the fork, any displacement of the particle will be repeated after every complete vibration of the fork and in this time, i.e., the time period of a complete vibration of the fork, the disturbance will travel outwards through a distance which is called wavelength.

The wave as described above is a *longitudinal progressive wave*, the vibrations of the particles of the medium being along the direction of propagation of the wave. If displacements of particles of the medium everywhere on a plane perpendicular to the direction of propagation are identical at any instant, the wave is a plane progressive wave.

 2. **Characteristics of a plane progressive wave of simple harmonic type :**

(1) Every particle describes simple harmonic motion along the direction of propagation of wave, there being a change of phase from point to point.

(2) The arrangement without changing its type advances with a uniform velocity, its value depending on the elastic constant and density of the medium.

(3) Any particular displacement at a particular instant is repeated at regular distances called wave lengths; the velocity and the acceleration of the particles of a wave length apart from one another are the same.

(4) If N is the number of vibrations per second of a particle at a point along the wave then $N\lambda = c$, where c is the velocity of the wave and λ = wave length.

3. **Equation of a plane progressive wave :** Consider a wave to be moving along the positive direction of x with a velocity c . Let the displacement at any instant t at $x=0$ be

$$\xi_1 = a \sin nt$$

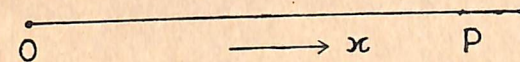


Fig. 22

To find the displacement at P at the same instant, we must remember that the disturbance has travelled without change of form from O to P with a velocity c . Hence displacement at O at time $(t - x/c)$ is the same as that at P at time t . Thus the motion at P will be given by

$$\xi = a \sin n(t - x/c) \quad \dots \quad (1)$$

If the wave moves towards the negative direction of the x -axis, the displacement at P at t will be the same as that at O after x/c seconds. Hence the displacement at P in this case will be

$$\xi = a \sin n(t + x/c) \quad \dots \quad (2)$$

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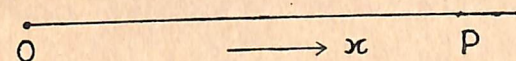


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$$\xi = a \sin n(t - x/c) \quad \dots \quad (1)$$

If the wave moves towards the negative direction of the x -axis, the displacement at P at t will be the same as that at O after x/c seconds. Hence the displacement at P in this case will be

$$\xi = a \sin n(t + x/c) \quad \dots \quad (2)$$

Since $n = 2\pi N = \frac{2\pi c}{\lambda}$

equation (1) can also be written as

$$\begin{aligned}\xi &= a \sin \frac{2\pi c}{\lambda} (t - x/c) \\ &= a \sin \frac{2\pi}{\lambda} (ct - x) \quad \dots (3)\end{aligned}$$

and (2) as

$$\xi = a \sin \frac{2\pi}{\lambda} (ct + x) \quad \dots (4)$$

4. Longitudinal plane progressive wave in an elastic medium :

Let $ABCD$ be a section of the elastic medium of area A , perpendicular to the direction of propagation of wave. Let $EFGH$ be a parallel section of equal area δx apart, i.e., $CG = \delta x$, δx being an elementary thickness of the layer.

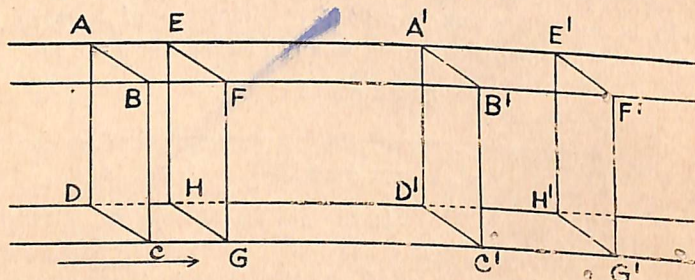


Fig. 23

Displacements of all particles on a plane perpendicular to the direction of propagation are same at any instant. Due to disturbance, let the particles on area $ABCD$ be displaced to ξ at any instant, so that $A'B'C'D'$ is the new displaced position; similarly particles on $EFGH$ are displaced to the new plane represented by $E'F'G'H'$.

Let co-ordinates of C and G in the initial positions be x and $x + \delta x$ respectively. Then those of C' and G' will be $x + \xi$ and $x + \xi + \delta x + \frac{d\xi}{dx} \delta x$ respectively.

Hence $C'G' = \left(x + \xi + \delta x + \frac{d\xi}{dx} \delta x \right) - (x + \xi) = \delta x + \frac{d\xi}{dx} \delta x$

Thus the increase in the thickness of the layer is $\frac{d\xi}{dx} \delta x$ and if there is no motion perpendicular to the direction of propagation, the increase in volume is $A \frac{d\xi}{dx} \delta x$ where A is the area of $ABCD$ or $EFGH$.

Hence volume strain $= \frac{A \frac{d\xi}{dx} \delta x}{A \delta x} = \frac{d\xi}{dx}$

and the excess pressure causing this strain will be given by

$$\delta p = -k \frac{d\xi}{dx}$$

where δp is the excess pressure over normal undisturbed pressure and k the modulus of elasticity.

Consider now the equilibrium of a layer of thickness δx of a section $ABCD$ of area A perpendicular to the wave : If $\frac{d\xi}{dx}$

is negative, i.e., there is a compression at A , there will be a force on area $ABCD$ perpendicular to it due to the medium on its left-hand side and this force on $ABCD$ towards the right is $-Ak \frac{d\xi}{dx}$. Similarly the force on $A'B'C'D'$ by the medium on the right hand side is

$$\begin{aligned}& -Ak \left(\frac{d\xi}{dx} + \frac{d}{dx} \frac{d\xi}{dx} \delta x \right) \\ &= -Ak \left(\frac{d\xi}{dx} + \frac{d^2 \xi}{dx^2} \delta x \right).\end{aligned}$$

This force is in the direction right to left on $A'B'C'D'$ due to medium on the right-hand side of $A'B'C'D'$.

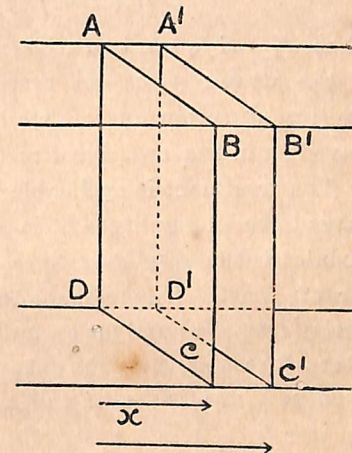


Fig. 24

Hence the resultant force on the layer in the positive direction of x is

$$-Ak \frac{d\xi}{dx} - \left\{ -Ak \left(\frac{d\xi}{dx} + \frac{d^2\xi}{dx^2} \delta x \right) \right\}$$

$$= Ak \frac{d^2\xi}{dx^2} \delta x.$$

If ρ_0 is the density of the layer, then the resultant force will be equal to $\rho_0 \delta x \frac{d^2\xi}{dt^2}$.

Hence

$$\rho_0 \frac{d^2\xi}{dt^2} = k \frac{d^2\xi}{dx^2}$$

$$\text{Or, } \frac{d^2\xi}{dt^2} = \frac{k}{\rho_0} \frac{d^2\xi}{dx^2}$$

$$\text{Or, } \frac{d^2\xi}{dt^2} = c^2 \frac{d^2\xi}{dx^2} \quad \dots (5)$$

where $c^2 = \frac{k}{\rho_0}$

Solution of equation (5) is given by

$$\xi = f_1(ct - x) + f_2(ct + x)$$

where f_1 and f_2 are two arbitrary functions and c is the velocity of the wave; $f_1(ct - x)$ represents a wave travelling in the positive direction of x , while $f_2(ct + x)$ represents a wave travelling in the negative direction.

The treatment is applicable to all cases of plane progressive waves of small amplitude. In a liquid or a gas, the appropriate modulus is the *adiabatic bulk modulus*. In the case of a solid bar of small transverse dimensions, the modulus is the *Young's modulus* and for an unlimited solid medium where large lateral forces are brought into play the modulus is the *axial modulus*. The cases of solids are treated separately in this chapter.

5. Acoustic pressure : Consider a simple harmonic wave in a gas travelling in the positive direction of x with a velocity c . The displacement at x is given by

$$\xi = a \sin \frac{2\pi}{\lambda} (ct - x)$$

and the excess pressure $\delta p = -k \frac{d\xi}{dx}$

$$= \frac{2\pi ak}{\lambda} \cos \frac{2\pi}{\lambda} (ct - x)$$

Hence the maximum excess pressure

$$\delta p_{max} = \frac{2\pi ak}{\lambda} \quad \dots (6)$$

This is known as acoustic pressure. Sometimes acoustic pressure is expressed in its root mean squared value. The R. M. S. acoustic pressure can be calculated as follows.

$$\text{Mean squared pressure} = \frac{1}{T} \int_0^T \left(\frac{2\pi ak}{\lambda} \right)^2 \cos^2 \frac{2\pi}{\lambda} (ct - x) dt$$

$$= \frac{1}{T} \left(\frac{2\pi ak}{\lambda} \right)^2 \cdot \frac{T}{2} = \frac{1}{2} \left(\frac{2\pi ak}{\lambda} \right)^2$$

$$\therefore \text{R. M. S. acoustic pressure} = \sqrt{2} \cdot \frac{\pi ak}{\lambda} \quad \dots (7)$$

6. Relation between displacement and excess pressure :

Let t be such that, $\frac{2\pi ct}{\lambda} = 2\pi s$ where s is an integer, then at that instant, from (3) and (6)

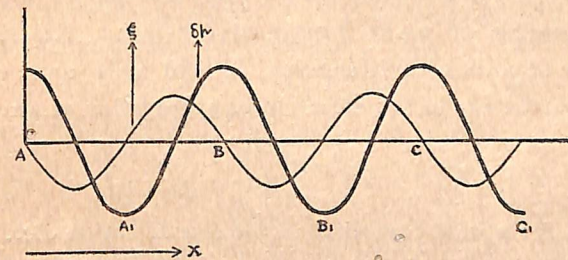


Fig. 25

$$\xi = -a \sin \frac{2\pi}{\lambda} x \quad \dots (8)$$

$$\text{and } \delta p = \frac{2\pi ak}{\lambda} \cos \left[\frac{2\pi}{\lambda} (-x) \right]$$

$$= \frac{2\pi ak}{\lambda} \cos \frac{2\pi x}{\lambda} \quad \dots (9)$$

Plotting ξ and δp with x , we get the above curves.

Hence the resultant force on the layer in the positive direction of x is

$$\begin{aligned} & -Ak \frac{d\xi}{dx} - \left\{ -Ak \left(\frac{d\xi}{dx} + \frac{d^2\xi}{dx^2} \delta x \right) \right\} \\ & = Ak \frac{d^2\xi}{dx^2} \delta x. \end{aligned}$$

If ρ_0 is the density of the layer, then the resultant force will be equal to $\rho_0 A \delta x \frac{d^2\xi}{dt^2}$.

Hence

$$\begin{aligned} \rho_0 \frac{d^2\xi}{dt^2} &= k \frac{d^2\xi}{dx^2} \\ \text{Or, } \frac{d^2\xi}{dt^2} &= \frac{k}{\rho_0} \frac{d^2\xi}{dx^2} \\ \text{Or, } \frac{d^2\xi}{dt^2} &= c^2 \frac{d^2\xi}{dx^2} \quad \dots (5) \end{aligned}$$

where $c^2 = \frac{k}{\rho_0}$

Solution of equation (5) is given by

$$\xi = f_1(ct - x) + f_2(ct + x)$$

where f_1 and f_2 are two arbitrary functions and c is the velocity of the wave; $f_1(ct - x)$ represents a wave travelling in the positive direction of x , while $f_2(ct + x)$ represents a wave travelling in the negative direction.

The treatment is applicable to all cases of plane progressive waves of small amplitude. In a liquid or a gas, the appropriate modulus is the *adiabatic bulk modulus*. In the case of a solid bar of small transverse dimensions, the modulus is the *Young's modulus* and for an unlimited solid medium where large lateral forces are brought into play the modulus is the *axial modulus*. The cases of solids are treated separately in this chapter.

5. Acoustic pressure : Consider a simple harmonic wave in a gas travelling in the positive direction of x with a velocity c . The displacement at x is given by

$$\xi = a \sin \frac{2\pi}{\lambda} (ct - x)$$

and the excess pressure $\delta p = -k \frac{d\xi}{dx}$

$$= \frac{2\pi ak}{\lambda} \cos \frac{2\pi}{\lambda} (ct - x)$$

Hence the maximum excess pressure

$$\delta p_{max} = \frac{2\pi ak}{\lambda} \quad \dots (6)$$

This is known as acoustic pressure. Sometimes acoustic pressure is expressed in its root mean squared value. The R. M. S. acoustic pressure can be calculated as follows.

$$\begin{aligned} \text{Mean squared pressure} &= \frac{1}{T} \int_0^T \left(\frac{2\pi ak}{\lambda} \right)^2 \cos^2 \frac{2\pi}{\lambda} (ct - x) dt \\ &= \frac{1}{T} \left(\frac{2\pi ak}{\lambda} \right)^2 \cdot \frac{T}{2} = \frac{1}{2} \left(\frac{2\pi ak}{\lambda} \right)^2 \end{aligned}$$

$$\therefore \text{R. M. S. acoustic pressure} = \sqrt{2} \cdot \frac{\pi ak}{\lambda} \quad \dots (7)$$

6. Relation between displacement and excess pressure :

Let t be such that, $\frac{2\pi ct}{\lambda} = 2\pi s$ where s is an integer, then at that instant, from (3) and (6)

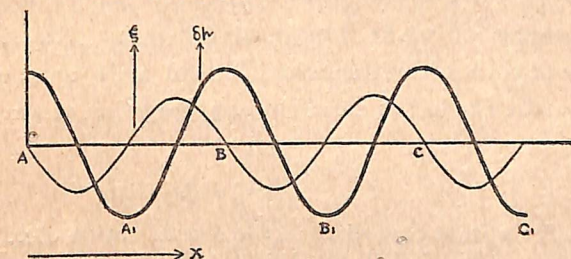


Fig. 25

$$\xi = -a \sin \frac{2\pi}{\lambda} x \quad \dots (8)$$

$$\begin{aligned} \text{and } \delta p &= \frac{2\pi ak}{\lambda} \cos \left[\frac{2\pi}{\lambda} (-x) \right] \\ &= \frac{2\pi ak}{\lambda} \cos \frac{2\pi x}{\lambda} \quad \dots (9) \end{aligned}$$

Plotting ξ and δp with x , we get the above curves.

From the curves we see that the pressure is normal undisturbed pressure at maximum and minimum displacements; it is maximum at points such as A, B, C etc., and minimum at A_1, B_1, C_1 etc., where the displacements are zero.

7. Bulk modulus of a gas and velocity of propagation :

When a sound wave passes through a gaseous medium, there are rapid pressure changes at any point in the path and the heat generated or absorbed has not sufficient time to be exchanged with the outside. The process is adiabatic and the gas law in adiabatic conditions is

$pv^\gamma = \text{constant}$, where γ is the ratio of two specific heats.

Differentiating,

$$v^\gamma \delta p + p \gamma v^{\gamma-1} \delta v = 0$$

$$\text{whence } \gamma p = -v \frac{dp}{dv}$$

and hence velocity of the wave is given by

$$c = \sqrt{\frac{k}{\rho_0}} = \sqrt{\frac{\gamma p}{\rho_0}} \quad \dots (10)$$

8. Energy of plane progressive waves : Kinetic energy of a layer of elementary thickness δx and unit cross-sectional area perpendicular to the direction of propagation of wave is

$$\frac{1}{2} \rho_0 \delta x \left(\frac{d\xi}{dt} \right)^2$$

Since for a wave travelling in the +ve direction of x .

$$\xi = a \sin \frac{2\pi}{\lambda} (ct - x)$$

K. E. of δx at any instant

$$= \frac{1}{2} \rho_0 \delta x \left[\frac{a \times 2\pi c}{\lambda} \cos \frac{2\pi}{\lambda} (ct - x) \right]^2$$

$$= \frac{1}{2} \rho_0 \delta x \cdot \frac{4\pi^2 a^2 c^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda} (ct - x).$$

$\therefore$ Average kinetic energy of the layer per vibration

$$= \rho_0 \delta x \frac{2\pi^2 c^2 a^2}{\lambda^2 T} \int_0^T \cos^2 \frac{2\pi}{\lambda} (ct - x) dt$$

$$= \rho_0 \delta x \frac{2\pi^2 c^2 a^2}{\lambda^2 T} \cdot T/2$$

$$= \rho_0 \delta x \frac{\pi^2 a^2 c^2}{\lambda^2} \quad \dots \dots (11)$$

The average potential energy per cycle of vibration can be calculated in the following way. If the excess pressure in the disturbed condition is δp , then average pressure is

$$\frac{p_0 + p_0 + \delta p}{2} = p_0 + \frac{\delta p}{2}$$

where p_0 is the pressure when the medium is undisturbed.

Since work done on a gas is $p \delta v$ where p is the pressure and δv is the change in volume, writing

$$\delta p = -k \frac{d\xi}{dx}$$

$$\text{and } \delta v = -\frac{d\xi}{dx} \delta x,$$

$$\begin{aligned} \text{work done} &= \left(p_0 - \frac{k}{2} \frac{d\xi}{dx} \right) \times -\frac{d\xi}{dx} \delta x \\ &= -p_0 \frac{d\xi}{dx} \delta x + \frac{k}{2} \left(\frac{d\xi}{dx} \right)^2 \delta x. \end{aligned}$$

Or, the average potential energy per cycle

$$= -\frac{p_0 \delta x}{T} \int_0^T \frac{d\xi}{dx} dt + \frac{k}{2T} \delta x \int_0^T \left(\frac{d\xi}{dx} \right)^2 dt.$$

Since $\frac{d\xi}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(ct-x)$

$$\therefore \int_0^T \frac{d\xi}{dx} dt = 0.$$

$\therefore$ Average potential energy of the layer δx per cycle

$$= \delta x \cdot \frac{k}{2T} \int_0^T \left(\frac{d\xi}{dx} \right)^2 dt$$

$$= \frac{\rho_0 c^2 \delta x}{2T} \int_0^T \frac{4\pi^2 a^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda}(ct-x) dt$$

$$= \frac{\rho_0 c^2}{2T} \frac{4\pi^2 a^2}{\lambda^2} \cdot \frac{T}{2} \delta x$$

$$= \rho_0 \delta x \frac{\pi^2 c^2 a^2}{\lambda^2}$$

... (12)

$\therefore$ Average kinetic energy is equal to average potential energy and the total energy of slice δx of unit cross-section

$$= \rho_0 \delta x \frac{2\pi^2 c^2 a^2}{\lambda^2}.$$

Hence energy density, i.e., energy per unit volume of the wave front

$$= 2\pi^2 \rho_0 \frac{c^2 a^2}{\lambda^2}.$$

... (13)

Now, intensity of sound energy at a point is the energy flowing per unit area perpendicular to the direction of propagation per unit time; hence it is the energy contained in a volume of unit cross-sectional area and of length c .

$$\therefore \text{Intensity} = \frac{2\pi^2 \rho_0 c^2 a^2}{\lambda^2} \times c$$

$$= \frac{2\pi^2 \rho_0 c^3 a^2}{\lambda^2}.$$

... (14)

Energy flow per unit area of the wave front can also be calculated in the following way.

At a point x along the wave

$$\delta p = -k \frac{d\xi}{dx}$$

$\therefore$ Rate of work done per unit area of the wave front

$$= \delta p \cdot \frac{d\xi}{dt}$$

$$= -k \frac{d\xi}{dx} \cdot \frac{d\xi}{dt}$$

Hence average work done,

$$w = -\frac{k}{T} \int_0^T \frac{d\xi}{dx} \cdot \frac{d\xi}{dt} \times dt$$

If $\xi = a \sin \frac{2\pi}{\lambda}(ct-x)$, then

$$\frac{d\xi}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(ct-x)$$

$$\text{and } \frac{d\xi}{dt} = \frac{2\pi ac}{\lambda} \cos \frac{2\pi}{\lambda}(ct-x)$$

$$\therefore w = \frac{4\pi^2 a^2 ck}{\lambda^2 T} \int_0^T \cos^2 \frac{2\pi}{\lambda}(ct-x) dt$$

$$= \frac{2\pi^2 a^2 ck}{\lambda^2} = \frac{2\pi^2 a^2 \rho_0 c^3}{\lambda^2}$$

9. Relation between acoustic pressure and energy density :

From (6), acoustic pressure

$$\delta p_{max} = \frac{2\pi ak}{\lambda} = \frac{2\pi a \rho_0 c^2}{\lambda}$$

$$\text{Or, } \frac{(\delta p_{max})^2}{\text{Intensity}} = \frac{4\pi^2 a^2 \rho_0^2 c^4}{\lambda^2} \times \frac{\lambda^2}{2\pi^2 \rho_0 c^3 a^2} = 2\rho_0 c$$

$\therefore$ Intensity or energy flow per second per unit area of wave front = $\frac{1}{2} \left[\frac{(\delta p_{max})^2}{\rho_0 c} \right]$... (15)

10. Plane progressive wave in a solid rod :

Let us take a bar whose length is large compared with the lateral dimension and consider transverse sections vibrate along the length.

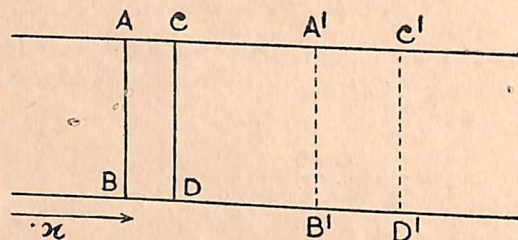


Fig. 26.

Let AB and CD be two perpendicular sections of the bar at x and $x+\delta x$ respectively where δx is an element of thickness. Suppose due to a longitudinal wave along the bar at any instant, the displacement of AB is ξ ; then that of CD at the same instance will be $\xi + \frac{d\xi}{dx}\delta x$. Hence the altered thickness $B'D'$ is $\delta x + \frac{d\xi}{dx}\delta x$.

Or, the longitudinal strain is $\frac{d\xi}{dx}\delta x/\delta x$.

Now consider the forces acting on a slice of the material of thickness δx and area A . Strain everywhere on the section $ABCD$ at x is $\frac{d\xi}{dx}$; hence the magnitude of force on area $ABCD$ due to the material of the bar lying on the left-hand side of $ABCD$ is $AE\frac{d\xi}{dx}$ where E is the Young's modulus of the material, this force being along the negative direction of x axis.

The force exerted on section $EFGH$ by portion of the bar lying on the right hand side of

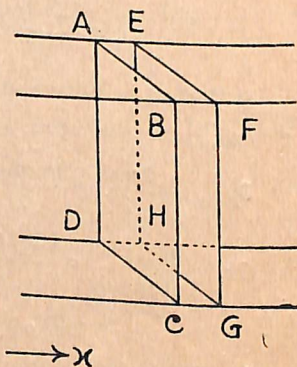


Fig. 27

$EFGH$ will be towards the positive direction of x and is equal to

$$AE\left(\frac{d\xi}{dx} + \frac{d}{dx}\frac{d\xi}{dx}\delta x\right) \\ = AE\frac{d\xi}{dx} + AE\frac{d^2\xi}{dx^2}\delta x.$$

Thus the resultant force on the element in the positive direction of x axis.

$$= \left(AE\frac{d^2\xi}{dx^2}\delta x + AE\frac{d\xi}{dx} \right) - AE\frac{d\xi}{dx} \\ = AE\frac{d^2\xi}{dx^2}\delta x.$$

This must be equal to the product of the mass of the element and its acceleration at the instant,

$$\text{or, } AE\frac{d^2\xi}{dx^2}\delta x = \rho\delta x\frac{d^2\xi}{dt^2}$$

$$\text{or, } \frac{d^2\xi}{dt^2} = \frac{E}{\rho}\frac{d^2\xi}{dx^2}$$

$$\text{or, } \frac{d^2\xi}{dt^2} = c^2\frac{d^2\xi}{dx^2} \quad \dots (16)$$

$$\text{where } c = \sqrt{E/\rho}$$

As in the previous case of a wave in a gaseous medium, the solution of the equation (16) is $\xi = f_1(ct-x) + f_2(ct+x)$ which represents two waves travelling along the positive and negative directions respectively with a velocity $c = \sqrt{E/\rho}$.

11. Velocity of sound in an unlimited solid medium :

In the foregoing article, we have assumed that lateral dimensions of the solid bar are small in comparison with its length. In an extended solid, the waves may assume transverse as well as longitudinal character and the vibrations are complex.

Let us consider the case of plane waves propagated through an infinite solid medium in a direction OX . In this case no lateral contraction or extension takes place and the appropriate

modulus will be the *axial modulus* X in place of Young's modulus E and the wave equation will be

$$\frac{d^2\xi}{dt^2} = \frac{X}{\rho} \frac{d^2\xi}{dx^2}$$

so that velocity of propagation C_1 is given by $C_1 = \sqrt{\frac{X}{\rho}}$.

Let us calculate the value of the axial modulus in terms of Young's modulus E and Poisson's ratio σ . Let us suppose stresses P, P_1 and P_1 are applied simultaneously along OX, OY, OZ respectively in the medium, OX being the direction of propagation of the waves. The strains along OX, OY, OZ are then $\frac{P}{E} - \sigma \frac{P'}{E} - \sigma \frac{P'}{E}$,

$$-\frac{\sigma P}{E} + \frac{P'}{E} - \frac{\sigma P'}{E} \text{ and } -\frac{\sigma P}{E} - \frac{\sigma P'}{E} + \frac{P'}{E} \text{ respectively.}$$

Since due to the presence of the medium no strain takes place along OY or OZ , $-\frac{\sigma P}{E} + \frac{P'}{E} - \frac{\sigma P'}{E} = 0$ whence $\sigma P = P'(1 - \sigma)$. Thus the stresses applied by the medium itself to prevent strain perpendicular to OX is $P' = P \cdot \frac{\sigma}{1 - \sigma}$.

But strain along OX is

$$\frac{P}{E} - \frac{2\sigma P'}{E} = \frac{P}{E} \left(1 - \frac{2\sigma^2}{1 - \sigma}\right) = \frac{P}{E} \left\{ \frac{(1 - 2\sigma)(1 + \sigma)}{(1 - \sigma)} \right\}$$

This strain is different from what it is in the absence of constraints perpendicular to the length as in the case of a thin rod.

We obtain the value of axial modulus as

$$X = \frac{E(1 - \sigma)}{(1 - 2\sigma)(1 + \sigma)}$$

The velocity of propagation C_1 of waves in an unlimited medium is then

$$C_1 = \sqrt{\frac{X}{\rho}} = \sqrt{\frac{E(1 - \sigma)}{(1 - 2\sigma)(1 + \sigma)\rho}}$$

If X is expressed in terms of bulk modulus k and rigidity modulus n , $X = \frac{3k + 4n}{3}$

$$\text{whence } C_1 = \sqrt{\frac{3k + 4n}{3\rho}}$$

If C is the velocity of propagation of longitudinal waves in a thin bar,

$$\frac{C_1}{C} = \sqrt{\frac{(1 - \sigma)}{(1 - 2\sigma)(1 + \sigma)}}$$

For $\sigma = .4$, $\frac{C_1}{C} = \sqrt{2.63}$ approximately and we see that $C_1 > C$.

If $\sigma = .5$, C_1 has an infinite value.

CHAPTER VII

STATIONARY WAVES

1. **Formation of stationary waves:** Two identical progressive waves travelling in opposite directions produce what are known as stationary waves. In this type of waves, the amplitude of vibration at any point x in the path of the component waves is a periodic function of x . At certain positions called **Nodes** there is no vibration at all, whereas at certain other points called **Antinodes**, vibration takes place with maximum amplitude.

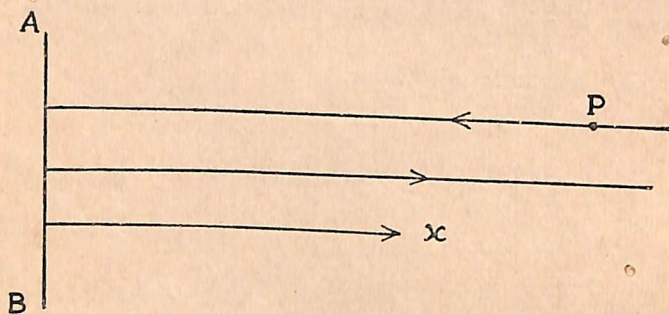


Fig. 28

Let two simple harmonic progressive waves of identical amplitude, period and magnitude of velocity travel in the same straight line, one in the positive direction and the other in the negative direction of x . Let the displacement at any instant of a particle of medium at $x=0$ due to the wave travelling along negative direction be $a \sin nt$ and that due to the wave travelling in the positive direction be $a \sin (nt + \delta)$. Then the displacement at a point P at a distance x due to the first wave is $\xi_1 = a \sin n\left(t + \frac{x}{c}\right)$ and that due to the second wave is

$$\xi_2 = a \sin \left\{ n\left(t - \frac{x}{c}\right) + \delta \right\}.$$

Hence the resultant displacement due to the two waves at P at time t is

$$\xi = \xi_1 + \xi_2 = 2a \cos \left(\frac{nx}{c} - \frac{\delta}{2} \right) \sin \left(nt + \frac{\delta}{2} \right) \quad \dots (1)$$

Thus the amplitude of vibration is a periodic function of x . The vibration will be maximum when the amplitude is $2a$ or $-2a$. The amplitude will be $2a$ at x_1 given by $\frac{nx_1}{c} - \frac{\delta}{2} = 2m\pi$, m being an integer. It will be $-2a$ at x_2 where $\frac{nx_2}{c} - \frac{\delta}{2} = (2m+1)\pi$. Thus positions of consecutive maximum vibrations are separated by a distance $x_2 - x_1 = \frac{c\pi}{n} = \frac{c\pi}{2\pi N} = \frac{\lambda}{2}$.

Again A will be zero at x_1' given by $\frac{nx_1'}{c} - \frac{\delta}{2} = (2m+1)\pi/2$. The next position of no vibration is given by

$$\frac{nx_2'}{c} - \delta/2 = (2m+3)\pi/2.$$

Hence positions of no vibrations are also separated by a distance $x_2' - x_1' = \frac{c\pi}{n} = \lambda/2$. Thus we see that successive antinodes or successive nodes are separated by a distance $\lambda/2$. Also it is apparent that the distance between an antinode and next node is half this distance, i. e., $\lambda/4$.

If AB is a rigid boundary and stationary waves are formed by the incident and the reflected waves, since the resultant displacement at $x=0$ is always zero, we have $A = 2a \cos \delta/2 = 0$ whence $\delta = \pi$.

Thus the incident wave suffers a phase change of π in its displacement after reflection. In this case we have as the expression for stationary waves in front of AB

$$\xi = 2a \sin \frac{nx}{c} \cos nt \quad \dots \dots (2)$$

putting $\delta = \pi$ in (1).

$$\text{or, } \xi = 2a \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi ct}{\lambda} \quad \dots \dots (3)$$

In the case discussed, we have considered that reflection takes place from a rigid boundary with a consequent phase change π . If however, there is no change of phase at reflection, the resultant wave will be given by

$$\begin{aligned}\xi &= a \sin \frac{2\pi}{\lambda}(ct+x) + a \sin \frac{2\pi}{\lambda}(ct-x) \\ &= 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi ct}{\lambda} \quad \dots \quad (4)\end{aligned}$$

$$= A_x \sin \frac{2\pi ct}{\lambda} \quad \dots \quad (5)$$

Here, there will be nodes when $\frac{2\pi x}{\lambda} = \pi/2, 3\pi/2, 5\pi/2$ etc., and antinodes when $\frac{2\pi x}{\lambda} = 0, \pi, 2\pi, 3\pi$ etc. The nodal or the antinodal distance will be the same, i.e., equal to $\lambda/2$.

If we plot amplitudes $A_x = 2a \sin \frac{2\pi}{\lambda} x$ and $2a \cos \frac{2\pi}{\lambda} x$ with $\frac{2\pi}{\lambda} x$, we shall get the curves for the two cases as given in fig. 29 & 30.

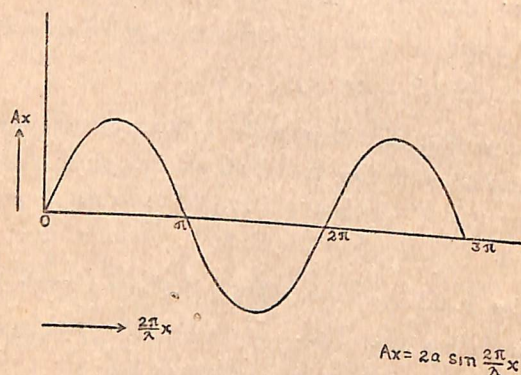


Fig. 29

Thus, if through one nodal distance the amplitude is positive, in the next one it becomes negative and vice versa. Hence the vibration is in the reverse direction after each node. Since $\xi = A_x \cos \frac{2\pi}{\lambda} ct$ or $\xi = A_x \sin \frac{2\pi}{\lambda} ct$ the phase of vibrations

remains same until A_x changes sign. Thus from one node to next node, the vibrations are in the same phase with changing

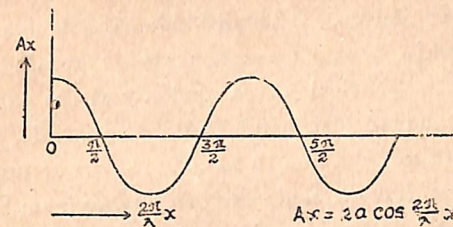


Fig. 30

amplitude; there is an abrupt change of phase by π in the vibrations throughout the next nodal distance.

2. Stationary waves in open and closed pipes: Let us consider what happens when a half wave of compression reaches the open end of a pipe. The front portion of the compression pushes air particles outside the opening and spreads out in the air outside in which maximum displacement without restraint is possible. As the front part of the compression has spread out with fall of pressure, the rear part of the compression in which there is increased pressure pushes out the air in front of it rapidly, causing a fall of pressure behind it. The air particles situated very near to the open end will move forward and this state of affairs will be transmitted backwards from layer to layer causing rarefaction to travel inwards from the open end. Hence a half-wave of compression is reflected back as a half-wave of rarefaction and a forward displacement of compression is reflected back as a forward displacement of rarefaction.

Thus the reflection at the open end is due to the openness of the medium outside the end. The greater the expansion of the compression, the more complete will be the reflection. The pressure at the open end is normal due to the joint effect of the incident and the reflected waves.

But in actual fact waves are propagated outside as spherical waves; otherwise no sound could be audible outside. Hence there must be some sort of density variation even at the mouth of the tube though less than what it is inside. As a consequence

there is a loss of energy and the reflected wave has a diminished amplitude, so that sound ultimately dies down unless maintained by an external source. The antinode cannot be exactly at the open end; then in that case plane waves would have to be transformed into spherical waves suddenly at the antinode. But since such a sudden discontinuity is not possible, there will be a gradual variation from one to the other type of waves and the spherical waves have a centre slightly away from the open end. Rayleigh from theoretical considerations found (in the case of a tube fitted with an infinite flange at the open end) that a correction term $0.824r$ must be added to the length of the tube where r is the radius of the tube.

In the case of reflection at a closed end, things are much simpler. Compression will be reflected back as compression and rarefaction as rarefaction. A forward displacement of air particles will be reflected by the rigid boundary as a backward displacement of equal magnitude. This is also evident from the fact that the resultant displacement at the closed end is always zero.

Let us suppose a tuning fork to be vibrating at the open end of a closed pipe. Let a wave of compression be transmitted towards the closed end by the vibration of the fork; this compression will be reflected back as compression from the closed end and in travelling outwards will be reflected at the open end as rarefaction which again will travel inwards, and striking at the closed end travel outwards as a rarefaction. Thus a complete wave of compression and rarefaction will be formed and if in the meantime the fork executes one complete vibration, there will be resonance and consequent radiation from the open end.

3. Stationary waves in pipes: The general wave equation is

$$\frac{d^2\xi}{dt^2} = \frac{k}{\rho} \frac{d^2\xi}{dx^2} = c^2 \frac{d^2\xi}{dx^2} \quad \dots \quad (6)$$

When stationary waves are formed, the amplitude at x is a function of x only. Hence we can put

$$\xi = f(x) \cos(nt + \epsilon) \text{ where } \epsilon \text{ is a phase term.}$$

$$\text{Then } \frac{d^2\xi}{dt^2} = -n^2 f(x) \cos(nt + \epsilon)$$

$$\frac{d^2\xi}{dx^2} = \cos(nt + \epsilon) \frac{d^2}{dx^2} f(x).$$

Substituting in (6)

$$-n^2 f(x) \cos(nt + \epsilon) = c^2 \cos(nt + \epsilon) \frac{d^2}{dx^2} f(x).$$

$$\text{Or, } \frac{d^2}{dx^2} f(x) + \frac{n^2}{c^2} f(x) = 0. \quad \dots \quad (7)$$

Solution of equation (7) is

$$f(x) = A \cos \frac{nx}{c} + B \sin \frac{nx}{c}, \quad A \text{ \& B being arbitrary}$$

constants.

$$\text{Hence } \xi = \left(A \cos \frac{nx}{c} + B \sin \frac{nx}{c} \right) \cos(nt + \epsilon). \quad \dots \quad (8)$$

which gives the general equation for stationary wave.

(i) Open pipes: If a pipe is open at both ends, there will be maximum vibrations and no density variation at the two ends, i. e., $\frac{d\xi}{dx} = 0$, (neglecting end errors) at $x=0$ and at $x=l$, where l is the length of the pipe.

$$\text{But } \frac{d\xi}{dx} = \left(-\frac{nA}{c} \sin \frac{nx}{c} + \frac{nB}{c} \cos \frac{nx}{c} \right) \cos(nt + \epsilon)$$

$$\text{When } x=0, \quad \frac{d\xi}{dx} = 0$$

$$\text{or, } 0 = \frac{nB}{c} \cos(nt + \epsilon).$$

Since $\cos(nt + \epsilon)$ is not zero for all values of t , B must be zero

$$\therefore \frac{d\xi}{dx} = -\frac{nA}{c} \sin \frac{nx}{c} \cos(nt + \epsilon).$$

But $\frac{d\xi}{dx} = 0$ at also $x = l$

$$\text{or, } 0 = -\frac{nA}{c} \sin \frac{nl}{c} \cos (nt + \epsilon)$$

$$\therefore \frac{nl}{c} = s\pi \text{ where } s = 1, 2, 3, \text{ etc.}$$

$$\text{or, } n = \frac{s\pi c}{l}.$$

But $n = 2\pi N$, where N is the frequency of vibration of the wave.

$$\text{Thus } 2\pi N = \frac{s\pi c}{l}$$

$$\text{or, } N = \frac{sc}{2l}.$$

Putting $s = 1, 2, 3$ etc., the possible modes of vibrations are given by $N = \frac{c}{2l}, \frac{2c}{2l}, \frac{3c}{2l}$ etc.

If a sound source emitting a note consisting of one or more of the above frequencies is placed at an open end, stationary waves will be produced and the open pipe will resonate.

(ii) **Closed pipes :** If the pipe is closed at $x = 0$, then ξ is zero for all values of t at that point.

From (8)

$$0 = A \cos (nt + \epsilon)$$

$$\therefore A = 0.$$

$$\text{Hence, } \xi = B \sin \frac{nx}{c} \cos (nt + \epsilon).$$

Now $\frac{d\xi}{dx} = 0$ at $x = l$ as density variation is negligible there.

$$\text{But } \frac{d\xi}{dx} = \frac{nB}{c} \cos \frac{nx}{c} \cos (nt + \epsilon)$$

$$\text{or, } 0 = \frac{nB}{c} \cos \frac{nl}{c} \cos (nt + \epsilon).$$

$$\text{Hence, } \cos \frac{nl}{c} = 0 \quad \text{or, } \frac{nl}{c} = \pi/2, 3\pi/2, 5\pi/2 \text{ etc.,}$$

$$\text{since } n = 2\pi N$$

$$\therefore N = \frac{c}{4l}, \frac{3c}{4l}, \frac{5c}{4l}, \dots \text{ etc.,}$$

which gives all the possible modes of stationary vibrations in a pipe closed at one end. Here, harmonics whose frequencies are even multiples of the fundamental (*i.e.*, $\frac{c}{4l}$) are not possible. Also the frequency of the fundamental in the case of a closed pipe is half that of the fundamental in an open pipe.

4. Energy of stationary waves : Suppose stationary waves are formed in a pipe closed at one end and the fundamental mode of vibration alone is excited. Then if the closed end is at $x = 0$, the displacement at a point x is

$$\xi = 2a \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi}{\lambda} ct. \quad \dots \quad \text{from (2)}$$

Now the energy of an element of volume $A\delta x$ of the waves at any instant is

$$E = \frac{1}{2} A \rho \delta x \left(\frac{d\xi}{dt} \right)^2 + \frac{1}{2} A \rho c^2 \left(\frac{d\xi}{dx} \right)^2 \delta x.$$

[where A = area of cross section of the pipe]

The first term gives the instantaneous kinetic energy and the second term the instantaneous potential energy.

Hence

$$E = \frac{A}{2} \rho \delta x \left(\frac{16\pi^2 a^2 c^2}{\lambda^2} \sin^2 \frac{2\pi x}{\lambda} \sin^2 \frac{2\pi ct}{\lambda} + \frac{16\pi^2 a^2 c^2}{\lambda^2} \cos^2 \frac{2\pi x}{\lambda} \cos^2 \frac{2\pi ct}{\lambda} \right) \dots \quad (9)$$

To find the average energy the expression is to be integrated with respect to t from 0 to T and then divided by T .

$$\text{Now } \frac{1}{T} \int_0^T \sin^2 \frac{2\pi ct}{\lambda} dt = \frac{1}{T} \int_0^T \cos^2 \frac{2\pi ct}{\lambda} dt = \frac{1}{2}$$

∴ Average energy of the wave per element of length δx is

$$E = A\rho\delta x \times \frac{4\pi^2 a^2 c^2}{\lambda^2} \left(\sin^2 \frac{2\pi x}{\lambda} + \cos^2 \frac{2\pi x}{\lambda} \right) \\ = A\rho\delta x \times \frac{4\pi^2 a^2 c^2}{\lambda^2}$$

Hence the energy per unit volume of the wave

$$= \frac{4\pi^2 a^2 c^2}{\lambda^2} \rho \quad \dots \quad (10)$$

At the nodes the kinetic energy is zero and the energy is wholly potential, whereas at the antinodes $\frac{d\xi}{dx}$ is always zero and the energy is wholly kinetic.

To calculate energy of the stationary wave for the fundamental in a closed pipe, expression (9) is to be integrated with respect to x from $x=0$ to $x=\lambda/4$

$$\text{Now} \quad \int_0^{\lambda/4} \sin^2 \frac{2\pi x}{\lambda} dx \\ = \frac{1}{2} \int_0^{\lambda/4} \left(1 - \cos \frac{4\pi x}{\lambda} \right) dx \\ = \frac{1}{2} \left(x - \frac{\lambda}{4\pi} \cos \frac{4\pi x}{\lambda} \right)_0^{\lambda/4} \\ = \lambda/8$$

and also

$$\int_0^{\lambda/4} \cos^2 \frac{2\pi x}{\lambda} dx \\ = \frac{1}{2} \int_0^{\lambda/4} \left(1 + \cos \frac{4\pi x}{\lambda} \right) dx \\ = \frac{\lambda}{8}$$

Hence the total energy of the stationary wave in the pipe

$$= \frac{A}{2} \rho \times \frac{16\pi^2 a^2 c^2}{\lambda^2} \cdot \frac{\lambda}{8} \left(\sin^2 \frac{2\pi ct}{\lambda} + \cos^2 \frac{2\pi ct}{\lambda} \right) \\ = \frac{A\rho\pi^2 a^2 c^2}{\lambda} \quad \dots \quad (11)$$

It can also be obtained from the expression (10) which gives energy per unit volume of the wave as $\frac{4\pi^2 a^2 c^2}{\lambda^2} \rho$. Thus the energy of volume $A\lambda/4$ is

$$\frac{4\pi^2 a^2 c^2}{\lambda^2} \rho \times A\lambda/4 \\ = A\rho \frac{\pi^2 a^2 c^2}{\lambda}$$

5. Energy flow across unit area perpendicular to the direction of component waves : As in the case of progressive waves, the rate of work done

$$w = \frac{1}{T} \int_0^T -k \frac{d\xi}{dx} \frac{d\xi}{dt} dt$$

Let us take $\xi = A \cos \frac{nx}{c} \cos (nt + \epsilon)$

$$\text{Then } w = -\frac{n^2 A^2 k}{c} \sin \frac{nx}{c} \cos \frac{nx}{c} \int_0^T \cos (nt + \epsilon) \sin (nt + \epsilon) dt \\ = 0,$$

Thus there is no transmission of energy through any area in the case of a stationary wave.

6. Longitudinal stationary waves in a solid rod : The wave equation for longitudinal vibration in a bar is as we have seen

$$\frac{d^2 \xi}{dx^2} = \frac{E}{\rho} \frac{d^2 \xi}{dx^2} = c^2 \frac{d^2 \xi}{dx^2}$$

Since for a stationary wave we can write

$$\xi = f(x) \cos (nt + \epsilon)$$

so that the displacement at any position x at any time t will be given by the expression (8)

$$i. e., \xi = \left(A \cos \frac{nx}{c} + B \sin \frac{nx}{c} \right) \cos (nt + \epsilon)$$

Case (1). Bar clamped at $x=0$ and free at $x=l$:

Here $\xi=0$ at $x=0$ for all values of t ,
whence $A=0$,

$$\therefore \xi = B \sin \frac{nx}{c} \cos (nt + \epsilon)$$

$$\text{Again } \frac{d\xi}{dx} = \frac{nB}{c} \cos \frac{nx}{c} \cos (nt + \epsilon)$$

Now $\frac{d\xi}{dx} = 0$ at $x=l$, as strain there is zero.

$$\therefore \cos \frac{nl}{c} = 0$$

$$\text{Or, } \frac{nl}{c} = \pi/2, 3\pi/2, \frac{5\pi}{2} \text{ etc.}$$

Putting $n=2\pi N$, where N is the frequency

$$N = \frac{c}{4l}, \frac{3c}{4l}, \frac{5c}{4l} \text{ etc.}$$

Thus the frequency of the fundamental tone in a bar fixed at one end and free at the other is

$$N = \frac{c}{4l} = \frac{1}{4l} \sqrt{\frac{E}{\rho}}$$

All the odd harmonics may be present.

Case (2). Bar free at both ends :

Here $\frac{d\xi}{dx} = 0$ at $x=0$ and $x=l$, strains being zero at those regions.

Proceeding in a similar manner as in the case of an open pipe,

$$N = \frac{c}{2l}, \frac{2c}{2l}, \frac{3c}{2l}, \frac{4c}{2l} \text{ etc.,}$$

i. e., all the harmonics may be present, the frequency of the fundamental being $N = \frac{c}{2l} = \frac{1}{2l} \sqrt{\frac{E}{\rho}}$, double that for a bar fixed at one end.

Case (3). Bar free at both ends but clamped at the middle :

If we take the clamped point as the origin, then $\xi=0$ at $x=0$ which gives $A=0$.

$$\text{Now } \frac{d\xi}{dx} = 0 \quad \text{at } x = \frac{l}{2}.$$

$$\text{which gives } \cos \frac{nl}{2c} = 0$$

$$\therefore \frac{nl}{2c} = \pi/2, 3\pi/2, 5\pi/2 \text{ etc.}$$

$$\text{Hence } N = \frac{c}{2l}, \frac{3c}{2l}, \frac{5c}{2l} \text{ etc.}$$

Thus the odd harmonics may be present, the frequency of the fundamental mode of vibration being $\frac{1}{2l} \sqrt{\frac{E}{\rho}}$.

7. Correction due to radius of a rod of circular section :

Let us take a rod free at both ends. The longitudinal displacement at any point x at an instant t can be written for the s th mode of vibration as

$$\xi = a \cos \frac{s\pi x}{l} \sin s nt$$

The kinetic energy of the rod at any instant t is, for the s th mode of longitudinal vibration is

$$\begin{aligned} \int_0^l \frac{1}{2} \rho A \left(\frac{d\xi}{dt} \right)^2 dx &= \frac{1}{2} \rho A \int_0^l \left(s n a \cos \frac{s\pi x}{l} \cos s nt \right)^2 dx \\ &= \frac{1}{2} \rho A \cdot s^2 n^2 a^2 \frac{l}{2} \cos^2 s nt, \end{aligned}$$

A being the cross section of the rod.

Now we know that a lateral strain is always accompanied with a longitudinal strain. Hence the transverse velocity at a point is $\frac{d\eta}{dt} = \frac{d}{dt} \left(\sigma r \frac{d\xi}{dx} \right)$ where σ is the Poisson's ratio and where $\frac{\eta}{r}$ is lateral strain.

Thus the kinetic energy of the rod due to transverse motion is

$$\begin{aligned} & \int_0^l \int_0^r \left(\frac{d\eta}{dt} \right)^2 \cdot dx \cdot \frac{1}{2} \rho \cdot 2\pi r dr \\ &= \frac{\sigma^2 \rho \pi r^4}{4} \int_0^l \left[\frac{d}{dt} \left(\frac{d\xi}{dx} \right) \right]^2 \cdot dx \\ &= \frac{\sigma^2 \rho \pi r^4}{4} \int_0^l \frac{n^2 a^2 \pi^2}{l^2} s^4 \cdot \sin^2 \frac{s\pi}{l} x \cos^2 s nt \cdot dx \\ &= \frac{\rho \pi r^4 \sigma^2}{4} \cdot \frac{n^2 a^2 \pi^2}{l^2} s^4 \cdot \frac{l}{2} \cos^2 s nt. \end{aligned}$$

Potential energy of the rod at any instant

$$\begin{aligned} & \frac{1}{2} EA \int_0^l \left(\frac{d\xi}{dx} \right)^2 \cdot dx \\ &= \frac{1}{2} EA \cdot \frac{a^2 s^2 \pi^2}{l^2} \cdot \frac{l}{2} \sin^2 s nt \end{aligned}$$

Since sum of kinetic and potential energies is constant

$$\begin{aligned} & \frac{1}{2} \rho A s^2 n^2 a^2 \cdot \frac{l}{2} \cos^2 s nt + \frac{1}{2} \cdot \frac{EA a^2 s^2 \pi^2}{l^2} \cdot \frac{l}{2} \sin^2 s nt \\ &+ \frac{\rho \pi r^4 \sigma^2}{4} \cdot \frac{n^2 a^2 \pi^2}{l^2} \cdot s^4 \cdot \frac{l}{2} \cos^2 s nt = \text{const.} \end{aligned}$$

Remembering $\pi r^2 = A$, we have after simplification

$$\rho n^2 \cos^2 s nt + \frac{E\pi^2}{l^2} \sin^2 s nt + \frac{\rho r^2 \sigma^2 \pi^2 s^2 n^2}{2l^2} \cos^2 s nt = \text{const.}$$

$$\text{or, } \rho n^2 \cos^2 s nt - \frac{E\pi^2}{l^2} \cos^2 s nt$$

$$+ \frac{E\pi^2}{l^2} + \frac{\rho r^2 \sigma^2 \pi^2 s^2 n^2}{2l^2} \cos^2 s nt = \text{const.}$$

$$\text{whence } \rho n^2 - \frac{E\pi^2}{l^2} + \frac{\rho r^2 \sigma^2 \pi^2 n^2 s^2}{2l^2} = 0$$

$$\therefore n^2 = \frac{\frac{E\pi^2}{l^2}}{\rho \left(\frac{r^2 \sigma^2 \pi^2 s^2}{2l^2} + 1 \right)}$$

$$\text{Writing } n = \frac{2\pi}{T'}$$

$$\begin{aligned} \text{we get } \frac{2\pi}{T'} &= \frac{\frac{\pi}{l} \sqrt{E}}{\sqrt{\rho} \sqrt{1 + \frac{r^2 \pi^2 s^2 \sigma^2}{2l^2}}} \\ &= \frac{2\pi}{T} \left(1 - \frac{r^2 \pi^2 s^2 \sigma^2}{4l^2} \right) \end{aligned}$$

$$\text{where } \frac{1}{T'} = \frac{\sqrt{\frac{E}{\rho}}}{2l}, \text{ the time period without correction.}$$

$$\text{or, } T' = T \left(1 + \frac{r^2 \pi^2 s^2 \sigma^2}{4l^2} \right)$$

Thus if the rod is vibrating in its fundamental mode, the time period is increased by a factor

$$1 + \frac{r^2 \pi^2 \sigma^2}{4l^2}.$$

For any other mode of vibration the time period of the fundamental will be changed to

$$T_0 = T \left(1 + \frac{r^2 \pi^2 s^2 \sigma^2}{4l^2} \right).$$

Hence the change will be greater for a higher value of s . For a small value of $\frac{r}{l}$ the change is, however, not appreciable.

TRANSVERSE VIBRATION OF STRINGS

1. Velocity of transverse waves in a string :

An ideal string is a perfectly flexible thin filament of wire of uniform diameter and mass uniformly spread over the length.

Let AB represent an element δx of a string without stiffness under a tension T lying along X -axis when undisturbed. Let A_1B_1 represent the element when the string is displaced in the

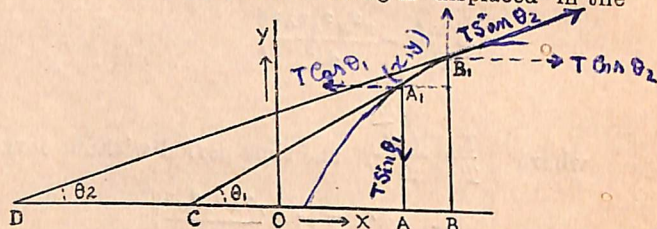


Fig. 31

plane XY . Let co-ordinates of A_1, B_1 be x, y and $x + \delta x, y + \delta y$ respectively. Let us suppose that the displacements are small and tension remains unaltered when the string vibrates in XY plane.

Let A_1C and B_1D be two tangents at A_1 and B_1 respectively making angles θ_1 and θ_2 with X -axis. Now the force on the element at A_1 and B_1 is T along A_1C and DB_1 respectively. Hence, the resultant force on the element along Y is $T \sin \theta_2 - T \sin \theta_1$

$$= T(\tan \theta_2 - \tan \theta_1) \quad [\because \theta_1 \text{ and } \theta_2 \text{ are small.}]$$

$$= T \left\{ \frac{dy}{dx} \left(y + \frac{dy}{dx} \delta x \right) - \frac{dy}{dx} \right\}$$

$$= T \frac{d^2 y}{dx^2} \delta x$$

$$\text{But the force on } \delta x = m \cdot \delta x \frac{d^2 y}{dt^2}$$

where m = mass per unit length of the string.

$$\therefore T \frac{d^2 y}{dx^2} \delta x = m \delta x \frac{d^2 y}{dt^2}$$

$$\text{Or, } \frac{d^2 y}{dt^2} = \frac{T}{m} \frac{d^2 y}{dx^2}$$

$$= c^2 \frac{d^2 y}{dx^2} \quad \dots (2)$$

$$\text{where } c = \sqrt{\frac{T}{m}}$$

The solution of equation (2) is

$$y = f_1(ct - x) + f_2(ct + x) \quad \dots (3)$$

which represents two transverse waves travelling along the positive and negative directions of X axis with a velocity $c = \sqrt{\frac{T}{m}}$

2. Stationary waves in a string :

Let the string be fixed very rigidly at $x=0$ and $x=l$; then the two waves will be reflected back from the supports and will interfere with each other.

Now $y=0$ at $x=0$ for all values of t . Hence

$$0 = f_1(ct) + f_2(ct)$$

$$\text{Thus } f_1(ct) = -f_2(ct).$$

Hence we can rewrite (3)

$$\text{as } y = f_1(ct - x) - f_1(ct + x).$$

Hence both the waves are of the same type and after reflection from the two ends will produce transverse stationary waves.

Again $y=0$ at $x=l$ for all values of t .

$$\therefore 0 = f_1(ct - l) - f_1(ct + l)$$

$$\text{Or, } f_1(ct - l) = f_1(ct + l)$$

Hence, the function is periodic whose value repeats after $2l$.

Let us suppose that due to formation of transverse stationary waves, any point of the string is vibrating simple harmonically, the amplitude of vibration being a function of x alone as in the case of all stationary waves.

Thus putting $y = f(x) \sin(nt + \epsilon)$, we get

$$y = \left(A \cos \frac{nx}{c} + B \sin \frac{nx}{c} \right) \sin(nt + \epsilon) \quad \dots \quad (4)$$

(See article 3, Page 80)

Now $y = 0$ at $x = 0$ for all values of t

$$\therefore 0 = A \sin(nt + \epsilon).$$

Hence $A = 0.$

Again $y = 0$ at $x = l$ for all values of t

$$\therefore 0 = B \sin \frac{nl}{c} \sin(nt + \epsilon)$$

which gives $\frac{nl}{c} = s\pi$, where $s = 1, 2, 3$ etc.

$$\text{or, } n = \frac{s\pi c}{l}$$

Thus all modes of vibrations with angular frequency $\frac{s\pi c}{l}$ and multiples of it are possible. Hence we can write

$$y = \sum_{s=1}^{s=\infty} B_s \sin \frac{s\pi x}{l} \sin \left(\frac{s\pi ct}{l} + \epsilon_s \right) \quad \dots \quad (5)$$

We can rewrite (5) as

$$y = \sum_{s=1}^{s=\infty} \left(a_s \cos \frac{s\pi ct}{l} + b_s \sin \frac{s\pi ct}{l} \right) \sin \frac{s\pi x}{l} \quad \dots \quad (6)$$

where a_s and b_s are new constants, such that $a_s = B_s \sin \epsilon_s$ and $b_s = B_s \cos \epsilon_s$.

Equation (6) gives all possible modes of vibrations for a uniform string under tension fixed at $x = 0$ and $x = l$, the frequency of the fundamental or the gravest mode of vibration being given by

$$2\pi N = \frac{\pi c}{l}$$

$$\text{or, } N = \frac{c}{2l} = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

the frequency of the s th mode is

$$N_s = \frac{sc}{2l} = \frac{s}{2l} \sqrt{\frac{T}{m}}$$

3. Energy of a vibrating string : If m = mass per unit length of a uniform string and if y the displacement of an element δx at x at any instant, then the kinetic energy of the element at that instant is $\frac{1}{2}m\delta x \left(\frac{dy}{dt} \right)^2$. Hence the kinetic energy of the whole string at an instant t is,

$$W = \frac{1}{2}m \int_0^l \left(\frac{dy}{dt} \right)^2 dx$$

To calculate potential energy of the element δx we can proceed in the following manner.

Let δs be the element in the displaced position. The work done against tension when the element is stretched from δx to δs is $T(\delta s - \delta x)$; or the potential energy of the element is $T(\delta s - \delta x)$.

From the figure $(\delta s)^2 = (\delta y)^2 + (\delta x)^2$

$$\text{whence } \delta s = \delta x \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \delta x \left\{ 1 + \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \right\}, \text{ neglecting higher}$$

order terms.

$$\text{Hence } T(\delta s - \delta x) = T\delta x \left\{ 1 + \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \right\} - T\delta x$$

$$= \frac{T}{2} \left(\frac{dy}{dx} \right)^2 \cdot \delta x$$

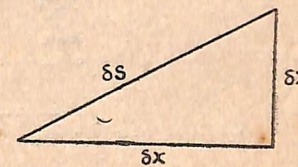


Fig. 32

Thus the potential energy of the whole string is

$$\begin{aligned}
 V &= \frac{T}{2} \int_0^l \left(\frac{dy}{dx} \right)^2 dx \\
 &= \frac{mc^2}{2} \int_0^l \left(\frac{dy}{dx} \right)^2 dx \quad \dots \quad (7) \\
 \therefore c &= \sqrt{\frac{T}{m}}
 \end{aligned}$$

$\therefore$ The total energy of the string at any instant is $W + V$

$$= \frac{m}{2} \int_0^l \left\{ \left(\frac{dy}{dt} \right)^2 + c^2 \left(\frac{dy}{dx} \right)^2 \right\} dx$$

Now the displacement at any instant at x , when the string is fixed at $x=0$ and $x=l$, is

$$y = \sum_{s=1}^{s=\infty} \left(a_s \cos \frac{s\pi ct}{l} + b_s \sin \frac{s\pi ct}{l} \right) \sin \frac{s\pi x}{l}$$

which can be re-written as

$$y = \sum_{s=1}^{s=\infty} A_s \sin \frac{s\pi x}{l} \cos \left(\frac{s\pi ct}{l} - \phi_s \right)$$

where $A_s = \sqrt{a_s^2 + b_s^2}$

and $\phi_s = \tan^{-1} \frac{b_s}{a_s}$

$$\text{Now } \frac{dy}{dt} = -\frac{\pi c}{l} \sum_{s=1}^{s=\infty} s A_s \sin \frac{s\pi x}{l} \sin \left(\frac{s\pi ct}{l} - \phi_s \right)$$

$$\begin{aligned}
 \therefore \left(\frac{dy}{dt} \right)^2 &= \left\{ \frac{\pi^2 c^2}{l^2} \cdot A_1^2 \sin^2 \frac{\pi x}{l} \sin^2 \left(\frac{\pi ct}{l} - \phi_1 \right) \right. \\
 &\quad + A_2^2 \cdot \frac{\pi^2 c^2}{l^2} \cdot 2^2 \cdot \sin^2 \frac{2\pi x}{l} \sin^2 \left(\frac{2\pi ct}{l} - \phi_2 \right) + \dots \\
 &\quad \left. + A_3^2 \cdot \frac{\pi^2 c^2}{l^2} \cdot s^2 \cdot \sin^2 \frac{s\pi x}{l} \sin^2 \left(\frac{s\pi ct}{l} - \phi_s \right) + \dots \right\}
 \end{aligned}$$

+ terms containing $\sin \frac{k_1 \pi x}{l} \times \sin \frac{k_2 \pi x}{l}$ as factors where k_1

and k_2 are integers, but $k_1 \neq k_2$.

$$\text{Now } \int_0^l \sin \frac{k_1 \pi x}{l} \cdot \sin \frac{k_2 \pi x}{l} dx$$

$$= \frac{1}{2} \int_0^l \left\{ \cos (k_1 - k_2) \cdot \frac{\pi x}{l} - \cos (k_1 + k_2) \cdot \frac{\pi x}{l} \right\} dx$$

$$= \frac{1}{2} \left[\frac{l}{\pi(k_1 - k_2)} \sin (k_1 - k_2) \cdot \frac{\pi x}{l} \right]_0^l$$

$$- \frac{1}{2} \left[\frac{l}{\pi(k_1 + k_2)} \sin (k_1 + k_2) \cdot \frac{\pi x}{l} \right]_0^l = 0$$

$$\therefore W = \frac{m\pi^2 c^2}{2l^2} \int_0^l \left\{ A_1^2 \sin^2 \frac{\pi x}{l} \sin^2 \left(\frac{\pi ct}{l} - \phi_1 \right) \right.$$

$$\left. + A_2^2 \cdot 2^2 \cdot \sin^2 \left(\frac{2\pi ct}{l} - \phi_2 \right) + \dots \right\} dx$$

$$\text{Now } \int_0^l \sin^2 \frac{s\pi x}{l} dx = \frac{1}{2} \int_0^l \left(1 - \cos \frac{2s\pi x}{l} \right) dx = \frac{l}{2}$$

$$\begin{aligned} \therefore W &= \frac{m\pi^2 c^2}{2l^2} \left\{ A_1^2 \frac{l}{2} \sin^2 \left(\frac{\pi ct}{l} - \phi_1 \right) + A_2^2 2^2 \cdot \frac{l}{2} \sin^2 \left(\frac{2\pi ct}{l} - \phi_2 \right) \right. \\ &\quad \left. + \dots + A_s^2 s^2 \cdot \frac{l}{2} \sin^2 \left(\frac{s\pi ct}{l} - \phi_s \right) + \dots \right\} \\ &= \frac{m\pi^2 c^2}{4l} \sum_{s=1}^{\infty} s^2 A_s^2 \sin^2 \left(\frac{s\pi ct}{l} - \phi_s \right) \quad \dots \quad (8) \end{aligned}$$

In a similar manner, we can work out the value of the potential energy at any instant.

$$\text{Potential energy } V = \frac{mc^2}{2} \int_0^l \left(\frac{dy}{dx} \right)^2 dx$$

$$\text{But } \frac{dy}{dx} = \frac{\pi}{l} \sum_{s=1}^{\infty} s A_s \cos \frac{s\pi x}{l} \cos \left(\frac{s\pi ct}{l} - \phi_s \right)$$

$$\therefore \left(\frac{dy}{dx} \right)^2 = \frac{\pi^2}{l^2} \sum_{s=1}^{\infty} s^2 A_s^2 \cos^2 \frac{s\pi x}{l} \cos^2 \left(\frac{s\pi ct}{l} - \phi_s \right)$$

+ terms having $\cos \frac{k_1 \pi x}{l} \cos \frac{k_2 \pi x}{l}$ as factors where k_1 and k_2

are integers but $k_1 \neq k_2$.

$$\begin{aligned} \text{But } \int_0^l \cos \frac{k_1 \pi x}{l} \cos \frac{k_2 \pi x}{l} dx \\ &= \frac{1}{2} \int_0^l \left\{ \cos \frac{\pi x}{l} (k_1 - k_2) + \cos \frac{\pi x}{l} (k_1 + k_2) \right\} dx \\ &= \frac{1}{2} \left[\frac{l}{\pi(k_1 - k_2)} \sin \frac{\pi x}{l} (k_1 - k_2) \right. \\ &\quad \left. + \frac{l}{\pi(k_1 + k_2)} \sin \frac{\pi x}{l} (k_1 + k_2) \right]_0^l = 0 \end{aligned}$$

$$\text{and } \int_0^l \cos^2 \frac{s\pi x}{l} dx = \frac{1}{2} \int_0^l \left(1 + \cos \frac{2s\pi x}{l} \right) dx = \frac{l}{2}$$

$$\therefore \text{Potential energy } V = \frac{m\pi^2 c^2}{4l} \sum_{s=1}^{\infty} s^2 A_s^2 \cos^2 \left(\frac{s\pi ct}{l} - \phi_s \right) \quad (9)$$

$\therefore$ The total energy of the vibrating string at any instant, $W + V$

$$\begin{aligned} &= \frac{m\pi^2 c^2}{4l} \sum_{s=1}^{\infty} s^2 A_s^2 \left\{ \sin^2 \left(\frac{s\pi ct}{l} - \phi_s \right) + \cos^2 \left(\frac{s\pi ct}{l} - \phi_s \right) \right\} \\ &= \frac{m\pi^2 c^2}{4l} \sum_{s=1}^{\infty} s^2 A_s^2 \quad \dots \quad (10) \end{aligned}$$

Remembering that the frequency of the s th mode $N_s = \frac{sc}{2l}$ and substituting the value of c in (10), we get total energy $W + V$

$$\begin{aligned} &= m\pi^2 l \sum_{s=1}^{\infty} A_s^2 \left(\frac{sc}{2l} \right)^2 \\ &= M\pi^2 \sum_{s=1}^{\infty} A_s^2 N_s^2 \quad \dots \quad (11) \end{aligned}$$

where $M = ml$, the mass of the string.

Thus the energy of vibration of a particular mode is proportional to the square of the frequency and the squares of the amplitude of that mode of vibration.

4. Plucked string

Let a string fixed at the ends be raised at some point of the string through some small distance perpendicular to the length, so that the two portions of the string form two sides of a triangle and then released. The string is said to be *plucked* and vibrations resulting thereof are due to *plucking*.

Now for a string fixed at $x=0$ and $x=l$, the most general solution of $\frac{d^2 y}{dt^2} = c^2 \frac{d^2 y}{dx^2}$ is as we have seen

$$y = \sum_{s=1}^{s=\infty} \left(a_s \cos \frac{s\pi ct}{l} + b_s \sin \frac{s\pi ct}{l} \right) \sin \frac{s\pi x}{l}$$

Let the string be plucked at $x=h$. Let k be the displacement at h at the instant of plucking.

Now $\frac{dy}{dt} = 0$ at $t=0$

But $\frac{dy}{dt} = \sum_{s=1}^{s=\infty} \left(-a_s \frac{s\pi c}{l} \sin \frac{s\pi ct}{l} + b_s \frac{s\pi c}{l} \cos \frac{s\pi ct}{l} \right) \sin \frac{s\pi x}{l}$

Hence at $t=0$

$$\begin{aligned} \frac{dy}{dt} = 0 &= \frac{\pi c}{l} \sum_{s=1}^{s=\infty} b_s \sin \frac{s\pi x}{l} \\ &= b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + b_2 \frac{2\pi c}{l} \sin \frac{2\pi x}{l} + \dots \\ &\quad + b_s \frac{s\pi c}{l} \sin \frac{s\pi x}{l} + \dots \end{aligned}$$

Since this is true for all values of x at $t=0$,

$$b_1 = b_2 = b_3 = \dots = 0$$

Thus for a plucked string, we get

$$\begin{aligned} y &= \sum_{s=1}^{s=\infty} a_s \sin \frac{s\pi x}{l} \cos \frac{s\pi ct}{l} \\ &= a_1 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + a_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi ct}{l} + \dots \\ &\quad + \dots + a_s \sin \frac{s\pi x}{l} \cos \frac{s\pi ct}{l} + \dots \end{aligned}$$

At $t=0$, $y=f(x)$ for any value of x is

$$\begin{aligned} y &= a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \dots \\ &\quad + \dots + a_s \sin \frac{s\pi x}{l} + \dots \end{aligned}$$

To evaluate a_s , multiply both sides by $\sin \frac{s\pi x}{l}$ and integrate with respect to x from $x=0$ to $x=l$.

Then as we have seen earlier

$$\int_0^l a_s \sin \frac{s'\pi x}{l} \times a_s \sin \frac{s\pi x}{l} dx = 0$$

when s' is an integer but $\neq s$.

Hence, $\int_0^l y \sin \frac{s\pi x}{l} dx = a_s \int_0^l \sin^2 \frac{s\pi x}{l} dx = a_s \frac{l}{2}$

Or, $a_s = \frac{2}{l} \int_0^l y \sin \frac{s\pi x}{l} dx$

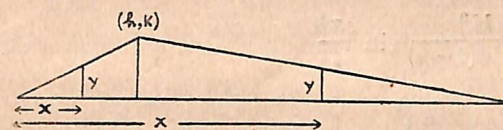


Fig. 33

But at $t=0$, $y = \frac{kx}{h}$, ($0 < x < h$)

and $y = \frac{k(l-x)}{(l-h)} = \frac{kl}{l-h} - \frac{k}{l-h}x$, ($h < x < l$)

Hence $\int_0^l y \sin \frac{s\pi x}{l} dx = \frac{k}{h} \int_0^h x \sin \frac{s\pi x}{l} dx$
 $+ \frac{kl}{l-h} \int_h^l \sin \frac{s\pi x}{l} dx - \frac{k}{l-h} \int_h^l x \sin \frac{s\pi x}{l} dx.$

But $\int x \sin \frac{s\pi x}{l} dx = -\frac{lx}{s\pi} \cos \frac{s\pi x}{l} + \int \frac{l}{s\pi} \cos \frac{s\pi x}{l} dx + \text{const.}$
 $= -\frac{lx}{s\pi} \cos \frac{s\pi x}{l} + \frac{l^2}{s^2\pi^2} \sin \frac{s\pi x}{l} + \text{const.}$

$$\begin{aligned}
\therefore \int_0^l y \sin \frac{s\pi x}{l} dx &= \frac{k}{h} \left[-\frac{lx}{s\pi} \cos \frac{s\pi x}{l} + \frac{l^2}{s^2\pi^2} \sin \frac{s\pi x}{l} \right]_0^h \\
&= \frac{kl}{(l-h)} \cdot \frac{l}{s\pi} \left[\cos \frac{s\pi x}{l} \right]_h^l - \frac{k}{l-h} \left[-\frac{lx}{s\pi} \cos \frac{s\pi x}{l} + \frac{l^2}{s^2\pi^2} \sin \frac{s\pi x}{l} \right]_0^l \\
&= \frac{k}{h} \left[-\frac{lh}{s\pi} \cos \frac{s\pi h}{l} + \frac{l^2}{s^2\pi^2} \sin \frac{s\pi h}{l} \right] - \frac{kl^2}{(l-h)s\pi} \left[\cos s\pi - \cos \frac{s\pi h}{l} \right] \\
&\quad - \frac{k}{l-h} \left[-\frac{l^2}{s\pi} \cos s\pi + \frac{lh}{s\pi} \cos \frac{s\pi h}{l} - \frac{l^2}{s^2\pi^2} \sin \frac{s\pi h}{l} \right] \\
&= \left[\frac{-kl}{s\pi} + \frac{kl^2}{(l-h)s\pi} - \frac{klh}{s\pi(l-h)} \right] \cos \frac{s\pi h}{l} \\
&\quad + \left[\frac{kl^2}{s^2\pi^2 h} + \frac{kl^2}{s^2\pi^2(l-h)} \right] \sin \frac{s\pi h}{l} \\
&= \frac{kl^3}{s^2\pi^2 h(l-h)} \sin \frac{s\pi h}{l} \\
\therefore a_s &= \frac{2}{l} \int_0^l y \sin \frac{s\pi x}{l} dx \\
&= \frac{2kl^2}{s^2\pi^2 h(l-h)} \sin \frac{s\pi h}{l} \quad \dots \quad (12)
\end{aligned}$$

Hence we can write

$$\begin{aligned}
y &= \frac{2kl^2}{\pi^2 h(l-h)} \left[\frac{1}{1^2} \cdot \sin \frac{\pi h}{l} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} \right. \\
&\quad + \frac{1}{2^2} \cdot \sin \frac{2\pi h}{l} \sin \frac{2\pi x}{l} \cos \frac{2\pi ct}{l} + \dots \\
&\quad \left. + \frac{1}{s^2} \cdot \sin \frac{s\pi h}{l} \sin \frac{s\pi x}{l} \cos \frac{s\pi ct}{l} \dots \dots \right] \quad \dots \quad (13)
\end{aligned}$$

Thus from the above relation we see that the s th harmonic vibration will disappear if $\sin \frac{s\pi h}{l} = 0$.

i.e., when $\frac{s\pi h}{l} = n\pi$ where n is any integer.

or when $s = \frac{nl}{h}$.

$h = l/2$, $s = 2n$. Putting $n = 1, 2, 3$ etc., we see that the 2nd, 4th, 6th and all the even harmonics will be absent. Similarly, 3rd, 6th, 9th etc., harmonics will disappear if $\frac{l}{h} = 3$. Thus if $\frac{l}{h} = p$ where p is an integer, p th, $2p$ th, $3p$ th etc., harmonics will not be present in the vibrations. In other words, we can say that if the string be plucked at a point where nodes of certain harmonics fall, then those harmonics will be absent. This is known as Young's Law.

The s th mode of vibration at point x from (13) is

$$y_s = \frac{A}{s^2} \cdot \sin \frac{s\pi h}{l} \sin \frac{s\pi x}{l} \cos \frac{s\pi ct}{l}$$

$$\text{where } A = \frac{2kl^2}{\pi^2 h(l-h)}.$$

At antinodes for this particular mode, $\frac{s\pi x}{l} = \pi/2$.

Hence the maximum amplitude of vibration of the s th mode is

$$\frac{A}{s^2} \cdot \sin \frac{s\pi h}{l}.$$

4. (a) Dependence of quality on the manner of plucking :

In the theory of the plucked string, it has been assumed that the string is plucked at a single point, so that $y = f(x)$ at $t = 0$ is a two step zigzag. It is actually so when the string is plucked by a sharp object like a steel wire or a plectrum. An infinite number of partials will be required to make up the initial wave and the note emitted will be very brilliant. If on the other hand, the string is plucked over a certain length by a round object or finger, the partials will be more rapidly convergent and the sound will be softer. In the extreme case, if the string be in the form of a sine wave, so that $y = A \sin \frac{\pi x}{l}$ at $t = 0$, then only the fundamental will be present in the subsequent vibrations.

4. (b) Effect of yielding of the bridge : The string is supposed to be fixed at $x = 0$ and $x = l$, so that there is no vibration beyond the supports. In practice, the bridge yields to

some extent, so that the vibration extends beyond it making ~~it~~ effective length greater than l . As a result the frequencies of all the partials are lowered in the same ratio and the note, therefore, still remains harmonic.

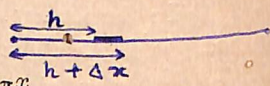
4. (c) Effect of stiffness of the wire :

The theoretical string is without stiffness. This is approximately true in the case of a very thin string. In an ideal string, the restoring force on an element is the component of tension acting opposite to the displacement. But if the string is thick, there is an additional elastic force due to bending. As a consequence, the partials no longer form a harmonic series. They are all raised in pitch, the higher partials by a greater proportion.

5. Struck String

The initial conditions of a plucked string could be described as static, while those of a *struck* string dynamic. Helmholtz and others considered that when a string is struck by a hammer, a sudden impulse is imparted to the part of the string struck, and initially velocity of all points on the string except the region struck is zero. The waves move in opposite directions from the struck point and are reflected at the fixed ends.

Let the string extend from $x=0$ to $x=l$. The displacement of such a string is given by

$$y = \sum_{s=1}^{s=\infty} \left(a_s \cos \frac{s\pi ct}{l} + b_s \sin \frac{s\pi ct}{l} \right) \sin \frac{s\pi x}{l}$$


Consider the string struck at an infinitesimally short region from $x=h$ to $x=h+\Delta x$. Let the instantaneous velocity imparted to the region at time $t=0$ be u .

Thus at $t=0$, $y=0$ and $\frac{dy}{dt}=0$ for all values of x except a short region Δx where it is u .

Since $y=0$ at $t=0$

$$\sum_{s=1}^{s=\infty} a_s \sin \frac{s\pi x}{l} = 0; \text{ hence } a_s = 0$$

$$\therefore y = \sum_{s=1}^{s=\infty} \sin \frac{s\pi ct}{l} \cdot b_s \sin \frac{s\pi x}{l} \quad \dots \quad (14)$$

$$\text{from which } \frac{dy}{dt} = \frac{\pi c}{l} \sum_{s=1}^{s=\infty} s b_s \sin \frac{s\pi x}{l} \cos \frac{s\pi ct}{l}$$

$$\text{At } t=0 \text{ we can write } \dot{y}_0 = \left(\frac{dy}{dt} \right)_{t=0} = \frac{\pi c}{l} \left(b_1 \sin \frac{\pi x}{l} + 2b_2 \sin \frac{2\pi x}{l} \dots + s b_s \sin \frac{s\pi x}{l} + \dots \right)$$

Applying Fourier's theorem

$$\begin{aligned} \frac{s\pi c b_s}{l} &= \frac{2}{l} \int_0^l \dot{y}_0 \sin \frac{s\pi x}{l} dx \\ &= \frac{2u}{l} \int_h^{h+\Delta x} \sin \frac{s\pi x}{l} dx, \quad (\because \dot{y}_0 = 0 \text{ except in the} \\ &\hspace{15em} \text{region from } h \text{ to } h+\Delta x) \\ &= -\frac{2u}{s\pi} \left\{ \cos \frac{s\pi}{l} (h+\Delta x) - \cos \frac{s\pi h}{l} \right\} \\ &= -\frac{2u}{s\pi} \left(\cos \frac{s\pi h}{l} \cos \frac{s\pi}{l} \Delta x - \sin \frac{s\pi h}{l} \sin \frac{s\pi \Delta x}{l} - \cos \frac{s\pi h}{l} \right) \\ &= \frac{2u \Delta x}{l} \sin \frac{s\pi h}{l} \\ &= \frac{2A}{l} \sin \frac{s\pi h}{l}, \text{ writing } u \Delta x = A \end{aligned}$$

$$\text{Hence } b_s = \frac{2A}{s\pi c} \sin \frac{s\pi h}{l} \quad \dots \quad (15)$$

$$\therefore y = \frac{2A}{\pi c} \sum_{s=1}^{s=\infty} \frac{1}{s} \sin \frac{s\pi h}{l} \sin \frac{s\pi x}{l} \sin \frac{s\pi ct}{l} \quad \dots \quad (16)$$

Thus the amplitude of the s th mode of vibration is inversely proportional to s . The series (16) is infinite and convergent, but less slowly convergent than in the case of a plucked string. Here also, if the point struck falls at one of the nodes of any possible mode of vibration, then that node cannot be generated by the

blow. Thus the truth of Young's Law is established by the theory.

In the above theory the time of contact of the hammer with the region struck has been supposed to be negligibly small compared with the periodic time of vibration. Kaufmann rejected the above theory, as experiments showed beyond doubt that time of contact is considerable in comparison with the periodic time.

Effect of touching the plucked or the struck point :

If after plucking or striking, the point is touched, all vibrations will cease; for those components which have not a node at the point are stopped by damping, whereas by Young's Law those components having a node at the point are absent from the very beginning.

6. Bowed string

Helmholtz, with the help of a vibration microscope, observed the motion of a violin string when bowed, and came to the following experimental laws.

(1) The vibration of all points on the string at any instant always takes place in a plane. The motion of any point on the string consists of an ascent with uniform velocity followed by a descent at another uniform backward velocity. For the middle point, the two velocities are equal. The displacement of a point on the string can be represented as a two step straight zigzag.

(2) At the place of bowing, the speed in the direction of bowing is equal to that of the bow.

Basing on these experimental laws Helmholtz gave a theoretical explanation of the vibration of a bowed string.

We know that the general solution for the displacement of a string fixed at $x=0$, and $x=l$, is

$$y = \sum_{s=1}^{s=\infty} (A_s \cos snt + B_s \sin snt) \sin \frac{s\pi x}{l} \quad \dots \quad (17)$$

$$\text{where } n = \frac{2\pi}{T} = \frac{\pi c}{l}$$

$$\therefore \frac{dy}{dt} = \sum_{s=1}^{s=\infty} (-sn A_s \sin snt + sn B_s \cos snt) \sin \frac{s\pi x}{l} \quad (18)$$

At the point observed have a constant forward velocity v_1 from $t=0$ to $t=T_1$ and a constant backward velocity v_2 from $t=T_1$ to $t=T$. By applying Fourier's theorem,

$$\begin{aligned} -sn A_s \sin \frac{s\pi x}{l} &= \frac{2}{T} \int_0^T \left(\frac{dy}{dt} \right) \sin snt \, dt \\ &= \frac{2}{T} \left(\int_0^{T_1} v_1 \sin snt \, dt + \int_{T_1}^T -v_2 \sin snt \, dt \right) \\ &= -\frac{1}{s\pi} \left(-(v_1 + v_2) + (v_1 + v_2) \cos sn T_1 \right) \\ &= \frac{2(v_1 + v_2)}{s\pi} \sin^2 \frac{ns T_1}{2} \quad \dots \quad (19) \end{aligned}$$

$$\text{Similarly, } sn B_s \sin \frac{s\pi x}{l} = \frac{2}{T} \int_0^T \left(\frac{dy}{dt} \right) \cos snt \, dt$$

$$\begin{aligned} &= \frac{(v_1 + v_2)}{s\pi} \sin sn T_1 \\ &= \frac{2(v_1 + v_2)}{s\pi} \sin \frac{sn T_1}{2} \cos \frac{sn T_1}{2} \quad \dots \quad (20) \end{aligned}$$

From (17), (19) and (20)

$$y = \frac{T(v_1 + v_2)}{\pi^2} \left\{ \sum_{s=1}^{s=\infty} \frac{1}{s^2} \sin \frac{sn T_1}{2} \sin sn \left(t - \frac{T_1}{2} \right) \right\} \quad (21)$$

Now (17) can be written as,

$$y = \sum_{s=1}^{s=\infty} c_s \sin \frac{s\pi x}{l} \sin sn(t - \alpha) \quad \dots \quad (22)$$

where α is a constant.

Comparing (21) and (22)

$$\begin{aligned} c_s \sin \frac{s\pi x}{l} &= \frac{T(v_1 + v_2)}{s^2 \pi^2} \sin \frac{sn T_1}{2} \\ &= \frac{T(v_1 + v_2)}{s^2 \pi^2} \sin \frac{s\pi T_1}{T} \quad \dots \quad (23) \end{aligned}$$

Consider at $x = \frac{l}{s}$, the contribution to the displacement of the s th harmonic vibration. It must be zero for all values of t . Since $v_1 + v_2$ is not zero, $\sin \frac{s\pi T_1}{T} = 0$. The possible solution is $\frac{T_1}{T} = \frac{1}{s} = \frac{x}{l}$ which is corroborated by the experimental fact that $\frac{T_1}{T} = \frac{1}{2}$ at the middle point and T_1 decreases with the decrease of x .

Now let us consider a transverse stationary wave for the fundamental vibration. If A is the amplitude at the middle point, then the amplitude at a point x is $A \sin \frac{\pi x}{l}$.

From (23) the amplitude at x is $\frac{T(v_1 + v_2)}{\pi^2} \sin \frac{\pi x}{l}$.

Thus we come to the conclusion that $\frac{T(v_1 + v_2)}{\pi^2}$ is constant or $v_1 + v_2$, the sum of the forward and backward velocities is a constant and is independent of x . At $x = \frac{l}{2}$, these two velocities are equal

$$\text{since } \frac{T_1}{T} = \frac{x}{l} = \frac{1}{2}.$$

Let a be the amplitude of vibration at $x = \frac{l}{2}$; then

$$v_1 + v_2 = \frac{2a}{T/2} + \frac{2a}{T/2} = \frac{8a}{T}.$$

Hence the equation (21) becomes

$$y = \frac{8a}{\pi^2} \left\{ \sum_{s=1}^{\infty} \frac{1}{s^2} \sin \frac{s\pi x}{l} \sin sn \left(t - \frac{xT}{2l} \right) \right\} \quad \dots (24)$$

The above theory of Helmholtz is based on the observation that the motion of a point on the bowed string is a two step zigzag. It is actually so when a violinist plays on his instrument bowing at the normal position at about $\frac{1}{10}$ th to $\frac{1}{15}$ th of the

whole length. But when bowed at other positions, the displacement curve sometimes deviates from the two step zigzag and the theory fails there. Moreover, there is another weakness of the theory. Young's Law is also true in the case of a bowed string. If the bowed point is at the node of one of the overtones then that tone cannot be generated by bowing. Helmholtz's theory contains no factor depending on the position of the bowed point.

6. (a) Raman's analysis of the bowed string :

Sir C. V. Raman explained that the experimental displacement curve could be obtained by the superposition of two "velocity waves" of wave length $2l$, moving in opposite directions. Velocity wave is obtained by taking velocity as ordinate and length of the string as abscissa. Such a velocity wave will be given by parallel straight lines with intervening discontinuities. If the two velocity waves in opposite directions move with a velocity $\sqrt{\frac{T}{m}}$, then from the resultant velocity diagram it is possible to explain all the experimental observations of Helmholtz, Krigar Menzel and others.

6. (b) Action of the bow :

The bow maintains the vibration of the string; the vibrations of the string can be supposed to be "free" as the period is not affected in the least by the bow. During part of the period, the resined hairs of the bow pull along with them the bowed part of the string and during this time the relative velocity between the bowed part and the bow is zero. Initially the string is straight; but as the bowed point is being dragged along, there is an increasing component of force due to tension in a direction opposite to the frictional force exerted by the bow. At some displacement, these two forces become equal and the string flies back and is caught again at some other position by the bow. According to Rayleigh, the bow can maintain the vibration due to the fact that solid friction is greater at smaller relative velocities.

When the bowed point moves with the bow, energy is supplied by it, and when it moves in the opposite direction,

the frictional force between the bow and the string opposes the motion of the bow. Let V_1, V_2 be the forward and backward velocities of the bowed point for interval T_1, T_2 respectively. If F is the force applied due to bowing, then supply of energy per cycle is

$$F\mu_s V_1 T_1 - F\mu_d V_2 T_2$$

when μ_s, μ_d are the coefficients of static and dynamic frictions respectively. If a is the amplitude at the bowed point then $\frac{a}{T_1} = V_1$ and $\frac{a}{T_2} = V_2$. Thus the supply of energy per cycle is $Fa(\mu_s - \mu_d)$.

6. (c) **Wolf note**: In instruments of violin family (bowed strings) and also in plucked instruments, it is sometimes seen that a particular note cannot be smoothly elicited. At this pitch the whole body of the instrument begins to vibrate and a howling effect is produced. Such a note is known as 'Wolf Note'. G. W. White came to the conclusion that the wolf note in a cello (violin family) coincided with the pitch of the best resonance of the belly of the cello.

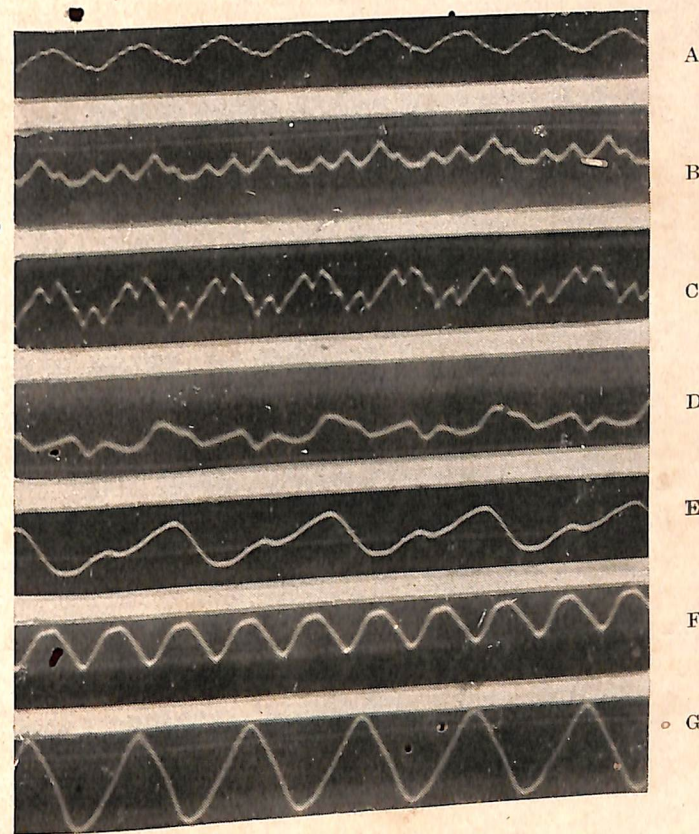
C. V. Raman explains wolf note in the following way. According to his theory a higher bowing pressure is needed for the fundamental than for the octave when the string is bowed at one end. At wolf pitch the body of the instrument vibrates with resonance and these vibrations take up so much energy from the bow that the fundamental can no longer be maintained and the vibration passes over into one in which the octave predominates.

7. Qualitative study of the stringed instruments :

The stringed instruments can be grouped in three types (1) Plucked, (2) Struck and (3) Bowed. Banjo, mandoline, guitar, harp etc., are the plucked western instruments whereas Sitar, Tanpura, Sarode, Vina etc., are the Indian ones of this type.

Again violin, viola etc., are bowed western instruments while Sarengi, Esraj etc., are the Indian counterparts. Piano is the only struck instrument and there is no similar Indian instrument.

VIBRATIONS OF STRINGS



A—SITAR.

B. C. & D.—TANPURA.

E. F. G.—CONVENTIONAL PLUCKED STRING.

The frequency of vibration N is given by

$$N = \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \cdot \beta^2$$

For the fundamental

$$N_1 = \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \times \frac{(1.88)^2}{l^2} \quad \dots \quad (6)$$

If N_1, N_2, N_3, N_4 etc., are the frequencies of the different modes of vibrations,

$$N_1 : N_2 : N_3 : \text{etc.}, :: (1.88)^2 : (4.69)^2 : (7.85)^2 \text{ etc.}$$

Thus the partials do not form a harmonic series as in the case of longitudinal vibration of bars or transverse vibration of strings.

(b) **Bar supported at both ends :** In this case

$$(i) \quad y = 0 \quad \text{at } x = 0$$

$$(ii) \quad \frac{d^2 y}{dx^2} = 0 \quad \text{at } x = 0$$

$$(iii) \quad y = 0 \quad \text{at } x = l$$

$$(iv) \quad \frac{d^2 y}{dx^2} = 0 \quad \text{at } x = l$$

From (i) and (ii)

$$A + C = 0$$

$$-A + C = 0 \quad \text{whence } A = 0 \quad \text{and } C = 0$$

$\therefore$ Equation (5) reduces to

$$y = (B \sin \beta x + D \sin h \beta x) \sin (nt - \delta) \quad \dots \quad (7)$$

From (iii) and (7)

$$B \sin \beta l + D \sin h \beta l = 0$$

From (iv)

$$-B \sin \beta l + D \sin h \beta l = 0.$$

$$\therefore \sin \beta l = 0.$$

$$\text{Or, } \beta l = s\pi \quad \text{where } s = 1, 2, 3 \text{ etc.}$$

$$\text{since } \beta^4 = \frac{n^2 \rho}{Ek^2}$$

$$n = k \sqrt{\frac{E}{\rho}} \times \beta^2$$

$$= \frac{k}{l^2} \sqrt{\frac{E}{\rho}} \cdot s^2 \pi$$

The frequency of vibration N is given by

$$N = \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \cdot \beta^2$$

For the fundamental

$$N_1 = \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \times \frac{(1.88)^2}{l^2} \dots \quad (6)$$

If N_1, N_2, N_3, N_4 etc., are the frequencies of the different modes of vibrations,

$$N_1 : N_2 : N_3 : \text{etc.}, :: (1.88)^2 : (4.69)^2 : (7.85)^2 \text{ etc.}$$

Thus the partials do not form a harmonic series as in the case of longitudinal vibration of bars or transverse vibration of strings.

(b) **Bar supported at both ends:** In this case

$$(i) \quad y=0 \quad \text{at } x=0$$

$$(ii) \quad \frac{d^2 y}{dx^2} = 0 \quad \text{at } x=0$$

$$(iii) \quad y=0 \quad \text{at } x=l$$

$$(iv) \quad \frac{d^2 y}{dx^2} = 0 \quad \text{at } x=l$$

From (i) and (ii)

$$A+C=0$$

$$-A+C=0 \quad \text{whence } A=0 \quad \text{and } C=0$$

$\therefore$ Equation (5) reduces to

$$y = (B \sin \beta x + D \sin h \beta x) \sin (nt - \delta) \dots \quad (7)$$

From (iii) and (7)

$$B \sin \beta l + D \sin h \beta l = 0$$

From (iv)

$$-B \sin \beta l + D \sin h \beta l = 0.$$

$$\therefore \sin \beta l = 0.$$

$$\text{Or, } \beta l = s\pi \quad \text{where } s = 1, 2, 3 \text{ etc.}$$

$$\text{since } \beta^4 = \frac{n^2 \rho}{Ek^2}$$

$$n = k \sqrt{\frac{E}{\rho}} \times \beta^2$$

$$= \frac{k}{l^2} \sqrt{\frac{E}{\rho}} \cdot s^2 \pi$$

Thus the frequencies are proportional to 1, 4, 9, 16 etc., and the partials are harmonic. This method is utilised in the construction of musical instruments consisting of bars of graded lengths, supported at ends. The bars when struck by hammers emit musical notes consisting of harmonic partials.

(c) Bar free at both ends :

Here, $\frac{d^2 y}{dx^2}$ and $\frac{d^3 y}{dx^3}$ are both equal to zero at $x=0$ and $x=l$. Thus the constants A, B, C, D can be found out.

Now from (5)

$$\left(\frac{d^2 y}{dx^2}\right) = (-A \cos \beta x - B \sin \beta x + C \cos h\beta x + D \sin h\beta x) \times \beta^2 \sin (nt - \delta)$$

$$\text{and } \left(\frac{d^3 y}{dx^3}\right) = (A \sin \beta x - B \cos \beta x + C \sin h\beta x + D \cos h\beta x) \times \beta^3 \sin (nt - \delta)$$

Since

$$\frac{d^2 y}{dx^2} \text{ and } \frac{d^3 y}{dx^3} = 0 \text{ at } x=0$$

we have $A=C$ and $B=D$.

From the conditions that $\frac{d^2 y}{dx^2}$ and $\frac{d^3 y}{dx^3} = 0$ at $x=l$

$$\begin{aligned} \text{we have } A (\cosh \beta l - \cos \beta l) &= B (\sin \beta l - \sinh \beta l) \\ \text{and } A (\sin \beta l + \sinh \beta l) &= -B (\cos \beta l - \cosh \beta l) \end{aligned}$$

Multiplying crosswise and simplifying, we have $\cosh \beta l = \sec \beta l$

... (8)

Plotting $y = \cosh \beta l$ and $y = \sec \beta l$ the points of intersection will give the required values of βl .

The values of βl are given below

$$| 4.73 | 7.85 | 10.996 | 14.137 | \text{etc.}$$

Now the frequency of the fundamental

$$N_1 = \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \cdot \frac{(4.73)^2}{l^2}$$

and those of partials of different modes are

$$\begin{aligned} N_2 &= \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \cdot \frac{(7.85)^2}{l^2} \\ N_3 &= \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \cdot \frac{(10.996)^2}{l^2} \end{aligned}$$

Thus $N_1 : N_2 : N_3 : N_4$ etc. :: 1 : 2.75 : 5.40 : 8.93 etc.

Hence the partials do not form a harmonic series.

(2) Tuning Fork : A tuning fork may be regarded (Chladni) as a free-free bar bent in the form of a U. If a free-free bar is gradually bent at the middle, it is seen that the nodes marked

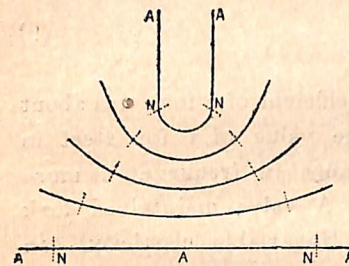


Fig. 35

as n approach each other as shown in the figure and the amplitude of vibration at the antinode at centre is small compared to that at either free end. The addition of a stem at the middle point causes the nodes to approach further. Due to the vibration of the prongs towards or away from each other their centres of gravity describe small arcs and consequently the stem has a small motion parallel to the prongs. When the stem is pressed on a sounding board, the energy of vibration of the fork is rapidly transmitted to it through the stem.

According to the second view (Rayleigh) each prong of a fork is regarded as a straight bar fixed at one end and free at the other. The pitch of the fundamental as deduced earlier on this approximation is

$$N = \frac{k}{2\pi} \sqrt{\frac{E}{\rho}} \cdot \frac{(1.88)^2}{l^2}$$

For a rectangular bar, the radius of gyration k about the neutral surface $= \frac{a}{\sqrt{12}}$ where a = thickness of the prong. For steel $\sqrt{E/\rho} = 5.24 \times 10^5$ cm/sec approximately.

Hence the pitch of the fundamental is

$$N = \frac{8.46 \times 10^4 a}{l^2}$$

2. (a) **Temperature variation of frequency**: If E_0, ρ_0, l_0 and E, ρ and l represent Young's modulus, density and length at 0°C and $t^\circ\text{C}$ respectively, then $E = E_0(1 - \gamma t)$, $l = l_0(1 + \alpha t)$ and $\rho = \rho_0(1 - 3\alpha t)$ where γ = temperature coefficient of Young's modulus and α = coefficient of linear expansion.

Hence substituting in (5) we have

$$N = \frac{k}{2\pi} \sqrt{\frac{E_0}{\rho_0}} \times \frac{(1.88)^2}{l_0^2} \times \frac{\left(1 - \frac{\gamma t}{2}\right)}{\left(1 - \frac{3\alpha t}{2}\right)(1 + 2\alpha t)}$$

$$= N_0 \left(1 - \frac{\gamma - \alpha}{2} t\right) \quad \dots \quad (9)$$

König found that temperature coefficient of a fork was about -11.2×10^{-5} per $^\circ\text{C}$. Substituting the value of α for steel in (9), it can be shown that the change in frequency is more dependent on change of E than on l . A valve maintained fork made of a steel alloy like elinvar (invariable elasticity) can maintain its frequency constant to within a few parts in a million when the temperature is kept constant by a thermostat.

2. (b) **Electrically maintained fork**: The vibration of a tuning fork may be maintained by an electromagnet.

M is an electromagnet, B a battery and C and D are electrical contact points. When the circuit is closed by a key, the current

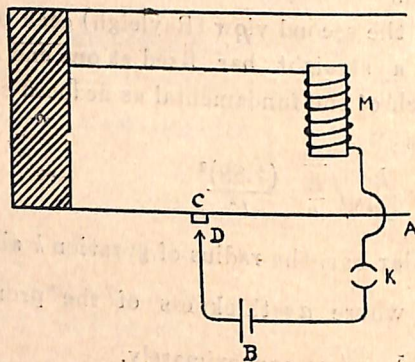


Fig. 36

passes into the coil through the contact points and the prongs are attracted by the electromagnet breaking the circuit. In a

more places and bowed with a violin bow at a point on one edge which is likely to be one theoretical position of an antinode. The vibration of the plate throws off the sand which ultimately collects along nodal lines. The plate used may be of glass, copper steel etc., but not of wood which has different elastic properties along different directions. Chladni preferred glass for his experiments. The plate may be held between the thumb and a finger or a hole may be drilled through it and then fixed by a screw and a nut. The plate must be touched at a position other than where it is fixed. For a good figure it may be necessary to shift the point of touch. Generally simple figures correspond to grave modes of vibration.

The following are a few of the simple figures which can be obtained easily on a square plate of size 50 cm. and thickness 2 cm. In these cases the plate is fixed at the centre, touched at A and bowed at B .

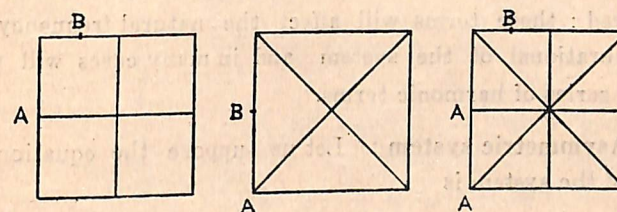


Fig. 41

Chladni's original work gives about fifty figures with square plates. It is also possible to have nodal figures with circular, elliptical or hexagonal plates.

CHAPTER XI

ASYMMETRIC VIBRATION : COMBINATOINAL TONES

1. **Large vibrations :** In the case of simple harmonic motions, the restoring force is proportional simply to displacement. But generally the restoring force may involve square, cube and other higher power terms of the displacement and we can write as equation of motion, neglecting damping as

$$\frac{d^2x}{dt^2} + n^2x + \alpha x^2 + \beta x^3 + \dots = 0 \quad \dots \quad (1)$$

where x is the displacement at any instant t and n, α, β etc., are constants. The constants α, β etc., are themselves very small and in the vibration where amplitude is small, $\alpha x^2, \beta x^3$ etc., can be neglected. But if vibrations are large, they can no longer be ignored; these terms will affect the natural frequency (at small vibrations) of the system and in many cases will give rise to a series of harmonic terms.

2. **Asymmetric system :** Let us suppose the equation of motion of the system is

$$\frac{d^2x}{dt^2} + n^2x + \alpha x^2 = 0 \quad \dots \quad (2)$$

Equation (2) represents an asymmetric vibration. The asymmetry is due to the term αx^2 in the restoring force. If the equation is solved by Rayleigh's method of successive approximation, we will have by an appropriate choice of initial conditions

$$x = a \cos pt + b \cos 2pt + c \cos 3pt + d \cos 4pt + e \quad \dots \quad (3)$$

where a, b, c, d, e are constant and p , the angular frequency of fundamental mode, is slightly smaller than n , the natural angular frequency of the system when α is zero. The amplitude a, b, c and d are progressively smaller. The presence of the term e represents asymmetry; it shows that the vibrations are displaced to one direction. A series of harmonic partials are present in the vibration.

3. **Symmetric vibrations :** In this case the equation of motion is

$$\frac{d^2x}{dt^2} + n^2x + \beta x^3 = 0$$

If we solve this equation by Rayleigh's method, we will see that the solution will be given by

$$x = a \cos pt + b \cos 3pt \quad \dots \quad (4)$$

Expression (4) shows that the vibration is symmetric about the position of rest. In addition to the fundamental vibration, there is the 3rd harmonic vibration. In this case also, p is slightly lower than n .

4. **Asymmetric system under double forcing :** Let us suppose an asymmetric system is subject to two separate simple harmonic forces. Neglecting damping, the equation of motion of such a system can be written as

$$m \frac{d^2x}{dt^2} + sx + kx^2 = F_1 \cos pt + F_2 \cos (qt + \theta)$$

where F_1 and F_2 are the amplitudes of the two simple harmonic forces of angular frequencies p and q respectively and θ represents the phase of the second harmonic force with respect to the first. Re-arranging.

$$\frac{d^2x}{dt^2} + n^2x + \alpha x^2 = A \cos pt + B \cos (qt + \theta) \quad \dots \quad (5)$$

where $n^2 = \frac{s}{m}, \alpha = \frac{k}{m}, A = \frac{F_1}{m}$ and $B = \frac{F_2}{m}$

Let us try to solve equation (5) by Rayleigh's successive approximation method. At first, let us neglect αx^2 ; then we get

$$\frac{d^2x}{dt^2} + n^2x = A \cos pt + B \cos (qt + \theta) \quad \dots \quad (6)$$

The solution of (6) is very simple and can be written as

$$x = a \cos pt + b \cos (qt + \theta) \quad \dots \quad (7)$$

The displacement x may be supposed to be equal to $x_1 + x_2$ where x_1 is the displacement due to the first force and x_2 due

to the second. The two displacements will be in phase with the two harmonic forces as frictional term is absent. Substituting the value of x as given in (7) in (5), we have

$$\frac{d^2x}{dt^2} + n^2x + \alpha[a \cos pt + b \cos (qt + \theta)]^2 = A \cos pt + B \cos (qt + \theta). \quad \dots \quad (8)$$

$$\begin{aligned} \text{But } \{a \cos pt + b \cos (qt + \theta)\}^2 &= a^2 \cos^2 pt + b^2 \cos^2 (qt + \theta) + 2ab \cos pt \cos (qt + \theta) \\ &= \frac{a^2}{2} (1 + \cos 2pt) + \frac{b^2}{2} \{1 + \cos 2(qt + \theta)\} \\ &\quad + ab \cos \{(p+q)t + \theta\} + \cos \{(p-q)t - \theta\}. \end{aligned}$$

$$\begin{aligned} \text{Hence } \frac{d^2x}{dt^2} + n^2x &= A \cos pt + B \cos (qt + \theta) - \frac{\alpha a^2}{2} \cos 2pt \\ &\quad - \frac{\alpha b^2}{2} \cos 2(qt + \theta) \\ &\quad - \alpha ab \cos \{(p+q)t + \theta\} - \alpha ab \cos \{(p-q)t - \theta\} \\ &\quad - \frac{\alpha(a^2 + b^2)}{2}. \quad \dots \quad (9) \end{aligned}$$

Thus we can regard the system to be under the influence of several simple harmonic forces acting simultaneously. The angular frequencies of the forces are p , q , $p+q$, $p-q$, $2p$ and $2q$. The solution of (9) will be obviously

$$\begin{aligned} x &= a \cos pt + b \cos (qt + \theta) + c \cos 2pt + d \cos 2(qt + \theta) \\ &\quad + e \cos \{(p+q)t + \theta\} + f \cos \{(p-q)t - \theta\} + g. \quad (10) \end{aligned}$$

Substituting the values of $\frac{d^2x}{dt^2}$ and n^2x in (9), we have

$$\begin{aligned} &a(n^2 - p^2) \cos pt + b(n^2 - q^2) \cos (qt + \theta) + c(n^2 - 4p^2) \cos 2pt \\ &+ d(n^2 - 4q^2) \cos 2(qt + \theta) + e\{n^2 - (p+q)^2\} \cos \{(p+q)t + \theta\} \\ &+ f\{n^2 - (p-q)^2\} \cos \{(p-q)t - \theta\} + n^2g \\ &= A \cos pt + B \cos (qt + \theta) - \frac{\alpha a^2}{2} \cos 2pt - \frac{\alpha b^2}{2} \cos 2(qt + \theta) \\ &\quad - \alpha ab \cos \{(p+q)t + \theta\} - \alpha ab \cos \{(p-q)t - \theta\} - \frac{\alpha(a^2 + b^2)}{2}. \end{aligned}$$

Equating from two sides

$$\begin{aligned} a &= \frac{A}{n^2 - p^2}, & b &= \frac{B}{n^2 - q^2} \\ c &= \frac{\alpha}{2} \left(\frac{a^2}{4p^2 - n^2} \right), & d &= \frac{\alpha}{2} \left(\frac{b^2}{4q^2 - n^2} \right) \\ e &= \frac{\alpha ab}{(p+q)^2 - n^2}, & f &= \frac{\alpha ab}{(p-q)^2 - n^2} \\ & \text{and } g &= -\frac{\alpha(a^2 + b^2)}{2n^2}. \end{aligned}$$

Thus we see the vibrations will consist of primary tones whose amplitudes are given by $\frac{A}{n^2 - p^2}$ and $\frac{B}{n^2 - q^2}$ respectively.

If the natural frequency is equal to the frequency of one of the impressed forces, then amplitude corresponding to that frequency will be very large. The amplitudes of the octaves will depend on the amplitudes a and b of the primary tones. Hence if a and b are small, these vibrations are negligible. Again we see that the amplitudes of vibration corresponding to $(p+q)$ and $(p-q)$ respectively are proportional to the product of those of the primaries. Hence these tones will be scarcely perceptible if the vibrations are small. There is another important point to note. The amplitude e of the summation tone of angular frequency $p+q$ is smaller in comparison with that of the difference tone of angular frequency $p-q$, as the denominator in the amplitude of the first tone is greater in comparison with that in the second tone.

To sum up, we see that an asymmetric system under the action of two simple harmonic forces of angular frequencies p and q respectively will have in the vibration of the system components corresponding to frequencies $2p$, $2q$, $p+q$ and $p-q$ in addition to the primary frequencies p and q . The additional vibrations other than the primary ones are negligible if amplitudes of the primaries are small.

5. Combination tones: When two tones are simultaneously sounded, an entirely new tone may be produced. The phenomenon was first observed by a German organist Sorge.

It was also known to the Italian violinist Tartini that two notes in the middle of the scale, a fifth apart, produce a tone whose pitch is the difference of the pitches of the two constituent notes. Such tones are the difference tones and were also called Tartini's tones after the name of the violinist. Helmholtz in course of his researches on the matter discovered another type of tones known as summation tones. The pitch of such a tone is the sum of the pitches of the generating tones. These difference and summation tones are together grouped as combination tones.

In order to make the combination tones audible, it is better to choose the resultant tone in the middle of the scale. To produce a difference tone, the generating tones should be of very high pitch, so that the difference tone is quite distinct from the pitches of the generators. It is rather difficult to produce the summation tone. In this case the generating tones should be of low pitch and probably a fifth apart, and the resultant summation tone should be in the middle of the scale. In producing the difference or the summation tone it is better to produce that note of the generator first which lies nearer to the resultant tone.

To produce a difference tone in the harmonium, first the note a' (853.4) is played and immediately afterwards c'' (1024). If the notes are played loudly together, the difference tone f (170.6) can be heard distinctly. Here the generating tones are in the treble of the scale. To produce summation tone by the same instrument, first the note c' (512) is played and then f' (341) is sounded along with c' . The resultant summation tone a' (853) will be heard.

6. Objective reality of combination tones :

König, Bosanquet and several other physicists believed that the combination tones are entirely subjective. They observed that the combination tones could not be reinforced by resonators. Helmholtz, on the other hand, showed that when two very intense notes are generated by blowing air from the same wind chest of his double siren through two series of holes, a very loud combination tone was produced whose objective

reality was unmistakably proved by a tuned resonator. He was of opinion that combination tones have sometimes objective reality. According to him the condition for objective reality was that the two primary tones must be very intense and produced by the same mass of air agitated very violently.

E. Edser and A. W. Rücker proved beyond doubt the objective existence of combination tones by a very delicate method. A mirror of Michelson's interferometer was attached to one prong of a tuning fork. Interference bands were produced when the prong was without movement, but at a slight vibration of the prong the bands disappeared from the field of view.

The generating tones were produced by a double siren through two series of holes and the pitches of the primaries were such that the difference or the summation tone if produced corresponded to the frequency of the tuning fork which was 64. The objective reality of the tone was proved beyond doubt by disappearances of fringes due to the vibration of the tuning fork in resonance with the pitch of the combination tone.

7. Theories of combination tones .

(a) Beat tone theory :

When two tones of nearly same the pitch are sounded, beats are produced recognised by periodic waxing and waning of sound with a frequency equal to the difference in the pitches of the component tones. (See Chapter I). If the difference in the frequencies be more than about 15 per second, the beats will be recognised as a separate tone. Such is the beat tone theory sometimes called König's theory. According to this view, the difference tone which is a beat tone cannot have any objective existence and, therefore, cannot be detected by a tuned resonator. Actually König's and Bosanquet's experiments confirmed this view. But beat tone theory is incapable of explaining summation tone discovered by Helmholtz, nor the theory can hold good when conclusive experiments prove objective reality of the combination tones.

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(b) Helmholtz's intensity theory :

Helmholtz's intensity theory is the theory of asymmetric vibrations of the system under the simultaneous influence of two large periodic forces. In this case the equation of motion is

$$\frac{d^2x}{dt^2} + n^2x + \alpha x^2 = F_1 \cos pt + F_2 \cos (qt + \theta)$$

We have seen earlier that the solution of the above equation is

$$x = a \cos pt + b \cos (qt + \theta) + c \cos 2pt + d \cos 2(qt + \theta) + e \cos \{(p+q)t + \theta\} + f \cos \{(p-q)t - \theta\} + g$$

Thus the above solution indicates the presence of difference and summation tones whose amplitudes are dependent on the product of the amplitudes of the primary tones represented by $a \cos pt$ and $b \cos (qt + \theta)$. Thus these tones will be produced when the harmonic forces are very intense. The asymmetry represented by αx^2 is responsible for the production of the combination tones and this term will have significance when x is very large.

(c) Waetzmann's general asymmetry theory :

Though Helmholtz's theory can explain most of the facts about combination tones, it is sometimes seen that combination tones can be heard even with very weak primaries. This can not be explained either by beat tone theory or the intensity theory. According to Waetzmann, these tones are produced in the ear drum itself which is a membrane loaded on one side. The natural vibration of a loaded membrane is asymmetric, that is, the vibration is displaced to one direction. If such a membrane is subjected to two weak harmonic forces, the general appearance of displacement curve will be like that in the formation of beats with the difference that the curve will lie above the line of zero displacement. By Fourier's analysis, Waetzmann showed that in addition to the primaries, the summation and difference tones are produced by the membrane. Waetzmann's theory thus combines and reconciles the beat tone theory and the intensity theory.

CHAPTER XII

DETERMINATION OF VELOCITY OF SOUND

1. Determination of velocity of sound in open air by signal method :

The earliest experiment on the velocity of sound in open air was made by Mersenne and Gassendi. The method consisted of a measurement of the time interval between noticing a flash of a gun-fire and receiving by the ear the report of the gun from a distance. The velocity is determined by dividing the distance between the gun and the receiving station by the time interval. This method is subject to two errors : (i) error due to wind velocity and (ii) error due to human element in the experiment or "personal equation". In the method recommended by the French Academy of sciences in 1738, the wind effect was eliminated by reciprocal firing and observations from two stations. The mean value of the velocity of sound reduced to 0°C was 332.2 metres per second.

Regnault tried to get rid of the human element or personal equation by registering the time interval between the instant of firing the gun and the reception of the report by electrical method. At the sending station a wire in the circuit of an electrical chronograph is broken by the shot. The circuit is re-established momentarily by a thin membrane when sound reaches the receiving station. The time interval is found out by two marks made by a style actuated by an electromagnet on a drum revolved at a constant speed by a clockwork. Regnault, however, came to the conclusion that lag in the recording instrument was of the same order as the personal equation of a trained observer.

Regnault came to the conclusion that velocity of sound increased with the intensity of the pulse.

2. Laboratory methods : (a) Hebb's method :

Accuracy of the measurements of sound velocity in the open air methods is subject to several factors, such as wind velocity,

personal equation, intensity of the wave and temperature of air in between two stations. Hence the results obtained by different experimenters vary. At the suggestion of A. Michelson, Hebb devised an objective laboratory method in measuring the velocity of sound in open air.

T_1 and T_2 are two carbon microphones placed at the foci of two co-axial paraboloid mirrors M_1 and M_2 made of plaster of Paris. A high pitched whistle of known frequency is placed

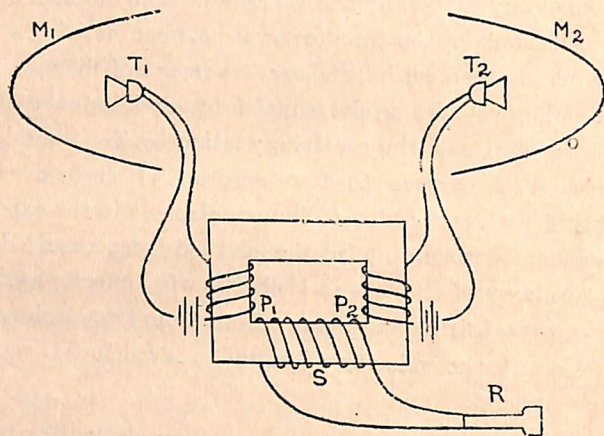


Fig. 42

at the focus of, say, M_1 . T_1 and T_2 are connected through batteries to two separate windings P_1 and P_2 of a special type transformer T having three windings. The telephone receiver is connected to the third winding.

Sound emitted by the whistle at the focus of M_1 after reflection from the two mirrors will be collected at the focus of M_2 . The resulting sound in the telephone will be the vector sum of sounds received by T_1 and T_2 . The receiving portion of the apparatus consisting of M_2 and T_2 is gradually shifted along the axis. At certain points sound will be reinforced and at other intermediate points it will be annulled depending on the phase difference of the two sounds collected at the two foci. The distance between two successive maxima or minima will be equal to the wave-length of the sound emitted by the source whose frequency is known. Thus velocity is determined.

Hebb's results gives the velocity of sound as 331.29 ± 0.04 metres per second at 0°C . The personal equation and uncertainties in the conditions of the medium are eliminated in this method.

3. Kundt's tube method :

It is one of the simplest methods of measuring velocity of sound in a gas or solid in the laboratory.

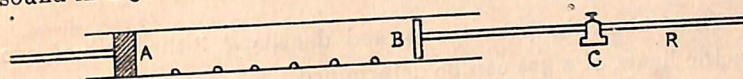


Fig. 43

A glass tube about 150 cm. long and about 5 cm. diameter has an adjustable stopper A at one end. Into the other end projects a disc B fitted to the end of a glass or metal rod clamped at the middle point C . The tube is thoroughly dried and fine lycopodium powder is sprinkled into it in small quantities. The metal rod is rubbed by a rosined cloth, so that longitudinal stationary waves appropriate to a rod clamped at the middle are produced in it. The fundamental tone which has the maximum amount of energy will have a wave-length $\lambda_s = 2l_s$ where l_s = the length of the rod. If c_s is the velocity of the longitudinal wave in the solid and N the frequency of vibration of the fundamental, then $c_s = 2l_s \times N$. The adjustable stopper A is gradually shifted till at some position the lycopodium powder in the tube collects in little heaps at some definite equidistant positions. In this condition, resonant stationary vibrations are set up in the air within the tube with production of nodes and antinodes, and lycopodium powder collects at nodes which are the positions of zero displacement. If l_a is the distance between two consecutive heaps of lycopodium powder, then c_a the velocity of longitudinal waves in air must be equal to $2l_a \times N$.

$$\therefore \frac{c_a}{c_s} = \frac{l_a}{l_s}$$

$$\text{or, } c_g = c_s \times \frac{l_a}{l_s} = \sqrt{\frac{E}{\rho}} \times \frac{l_a}{l_s}$$

where E and ρ are the Young's modulus and density of the solid rod.* Thus knowing E and ρ , the velocity in air can be determined.

4. Applications of Kundt's tube method :

The following are the applications of the Kundt's tube method.

(a) Measurement of velocity of sound in any gas :

We know that velocity of sound is given by

$$c = \sqrt{\frac{\gamma p}{\rho}}$$

Thus knowing pressure p and density ρ , γ the ratio of two specific heats of a gas can be determined.

(b) Determination of velocity of sound in a solid :

(c) Determination of dependence of velocity of sound on temperature, pressure and humidity :

(d) Rapid comparison of velocities of sound in two gases :

In this case the double-tube method is adopted. The same solid rod clamped at the middle projects into two separate tubes. Stationary vibrations are formed in both the tubes simultaneously after adjustment of the stoppers at the ends of each tube. The velocities of sound in the two tubes containing two different gases are in the ratio of the nodal distance in each tube. The velocities within the same gas at different condition of pressure and temperature etc., also can be rapidly compared.

The velocity of sound in a gas determined by Kundt's tube method must be corrected for the radius of the tube. Let us suppose v_1 and v_2 are the velocities determined with tubes of radii r_1 and r_2 respectively, then if v be the velocity in open air, we can write $v_1 = v(1 - k/r_1)$ and $v_2 = v(1 - k/r_2)$. The correction is zero with a tube of infinite radius. From the two expressions of the velocities we get

$$v = \frac{v_1 r_1 - v_2 r_2}{r_1 - r_2}$$

* See Chapter VI.

5. Striations in Kundt's tube : Antinodal discs :

When the vibrations are vigorous, in addition to the collection of dust particles in heaps at the nodes, there is a formation of striations consisting of particles across the bottom of the horizontal tube. The ridge like formations are more pronounced at the antinodes, disappearing altogether at the nodes. The phenomenon is more marked when the vibrations are intense.

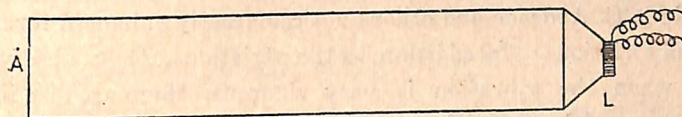


Fig. 44

König explained the formation of ridges from the hydrodynamical consideration of forces between two very small particles in a stream of air. He found that two small spheres of cork with the line joining their centres along a stream of air experience a repulsion ; if the line is perpendicular to the stream, there will be a force of attraction. König showed that the force of repulsion between two particles end on to a stream of air is

$$F_1 = 6\pi\rho r_1^3 r_2^3 \frac{v^2}{a^4}$$

and the force of attraction between them when broadside on to the stream is

$$F_2 = 3\pi\rho r_1^3 r_2^3 \frac{v^2}{a^4}$$

where ρ = density of air, r_1, r_2 the radii of the spherical particles, a distance between the centres and v^2 the mean squared velocity of the stream.

Thus two particles end on to the axis of the tube will be repelled from each other and when anyone of them happens to be near another particle, the line joining their centres being transverse to the axis, the two will be attracted to each other. This effect will be more pronounced near the antinode as v^2 of the air stream is maximum and will be absent at a node where it is zero.

More recently, investigators have used instead of a solid rod a loud speaker unit or a telephone diaphragm fed by the current of pure wave form from a variable frequency oscillator. The power supplied to the loud speaker unit may be as much as consistent with the construction of the speaker unit. The other end may be permanently closed. The frequency of the current into the coil of the loudspeaker is adjusted until resonant stationary vibrations are produced in the tube. Using such a unit, E. N. da C. Andrade and others noticed greatly enhanced formation of striations. In addition to the striations, they observed that when the vibration is very vigorous, there are disc like formations of dust particles across the tube of great sharpness and the distance between two such consecutive Antinodal discs can be measured with great precision to give the wave-length of the vibrations very accurately.

E. N. da C. Andrade has investigated motion of air inside a Kundt's tube in great details. By introducing very small particles like those of tobacco smoke, he observed that there is motion of air parallel to the axis of the tube from antinode to node and back again along the axis. Such motion was predicted by Lord Rayleigh and was experimentally verified by Andrade. In addition to the general circulation of air, there is a vortex motion round each dust particle. Andrade explained that the particles set them across the tube because of coalescing of the vortices round two nearby particles into one single vortex. The vigour of the vortex motion increases with the velocity of air stream and hence, the formation of ridges or antinodal discs are facilitated when the sound intensity is very large. The theory given by Konig as regards formation of striation seems to be wrong on the basis of the experiments of Andrade.

6. Velocity by resonant air column : This is a very simple laboratory method of determining the velocity of sound in air. A tuning fork vibrating at a frequency N is held near the open end of a tube whose other end dips into a column of water of variable height. The level of water column is adjusted till at some length l_1 of the tube above water surface, resonant vibrations are set up within the tube with a node at the closed end and an antinode at the open end. Taking into account the

fact that antinode occurs slightly away from the open end, we can write $\frac{\lambda}{4} = l_1 + kr$ where $k = .6$ is the correction for the open end.

If l_2 is the next resonant length obtained by adjusting the height of the water column, then together with an antinode at open end and a node at the closed end there will be another node and an antinode within the tube. In this case $\frac{3\lambda}{4} = l_2 + kr$.

From the two expressions, we get after eliminating end corrections $\frac{\lambda}{2} = l_2 - l_1$ and velocity of sound C as $C = N\lambda = 2N(l_2 - l_1)$.

7. The velocity of sound in a gas and the molecular velocity: The velocity of sound in a gas is given by $C = \sqrt{\frac{\gamma p}{\rho}}$. From kinetic theory of gases we know that $p = \frac{1}{3}\rho\bar{V}^2$ where $\bar{V}^2$ is the mean squared velocity of the molecules.

$$\text{Hence } C^2 = \frac{\gamma \rho \bar{V}^2}{3\rho} = \frac{\gamma}{3}\bar{V}^2.$$

If V_1 is the mean velocity of the molecules, we know from kinetic theory

$$\bar{V}^2 = \frac{3\pi}{8} V_1^2.$$

$$\therefore C^2 = \frac{\gamma}{3} \times \frac{3\pi}{8} V_1^2$$

$$\text{or, } C = V_1 \sqrt{\frac{\gamma\pi}{8}}.$$

Thus the velocity of sound in a gas and the molecular velocity are of the same order.

8. Velocity of sound in a liquid : The relation $c = \sqrt{\frac{k}{\rho}}$ is also applicable in the case of a liquid. Here k is the adiabatic elasticity. In most cases the adiabatic and isothermal elasticities of liquids differ by a very small amount. The velocity of sound in a liquid should be better written in the form $c = \sqrt{\frac{\gamma k}{\rho}}$ where k is the isothermal volume elasticity and γ = ratio of sp. hts.

2. Experimental determination of velocity of sound in liquids : Colladon and Sturm made a determination of velocity of sound in water in 1826 on lake Geneva. A hammer struck a bell in water which also flashed a quantity of gun powder above the water surface. An observer on the other side of the lake counted by means of a quarter second stopwatch the interval between the observation of the flash and the reception of the sound signal. The velocity of sound was found to be 1435 metres/second at 8°C and agrees excellently with the calculated value of 1436.4 metres/second.

M. Marti in 1919 made an under water determination of velocity of sound at a depth of 13 metres. Four hydrophones were placed in a straight line at a distance of 900 metres from each other. A charge of dry gun cotton in the same straight line as the hydrophones and at a distance of 1200 metres from the nearest hydrophone was exploded first on one side and then on the other. The reception of the sound in each hydrophone was recorded automatically in a chronograph. The mean value of velocity of sound at 15°C at a pressure of one atmosphere was 1504.15 metres/second.

The modern methods of determining velocity of sound in liquids by application of ultrasonic waves will be dealt with in a later chapter.

CHAPTER XIII

SOUND MEASUREMENT AND ANALYSIS

1. Measurement of frequency : (i) Absolute methods :

(a) We have seen in chapter XI how the frequency of a tuning fork may be measured with a great accuracy by phonic motor method or stroboscopic method. Several tuning forks measured by the above methods may be taken as laboratory substandards of frequency. If the frequency of a tone to be measured lies near that of one of the forks, then it can be found out by the method of beats.

(b) **Siren :** It consists of a disc with a ring of holes equidistant from the centre and rotating above another co-axial disc with similar holes. The lower fixed disc forms the top plate of a wind chest. In course of rotation of the upper disc, the two systems of holes coincide and a blast of air passes outside through the holes. The pitch of the sound emitted by the siren is then nm , where n = no. of equidistant holes and m = no. of revolutions of the disc per second.

The air blast is obtained from a reservoir, where pressure is maintained constantly by a compressor. If the two systems of holes are set at opposite obliquities along the circumference, the upper disc will rotate by reaction of the pulse of air escaping out of the holes. Or the disc may be rotated by an electric motor whose speed may be regulated. In any case, the revolutions per second are registered by a revolution counter.

By increasing the pressure of air or by regulating the speed of the motor, the upper disc of the siren is gradually speeded up till beats are formed between the note of the siren and the tone to be measured. By slightly increasing the speed of the siren it is very easy to understand which note has the higher frequency.

The accuracy in the above methods ultimately depends on the clock by which measurements are made.

(c) **Photographic method :** The tone may be received by a microphone and after proper amplification by a valve circuit the wave-form may be recorded on a photographic film crossed by standard time-marks.

(ii) **Comparative method:** A sonometer wire with a constant tension may be calibrated in frequencies by a set of tuning forks of graded frequencies which are again previously determined by the phonic motor method. The length of the wire is then adjusted till slow beats are obtained with the tone whose frequency is to be determined.

2. Analysis of frequency: Analysis of frequency can be conveniently made by means of a **Helmholtz Resonator**.

A Helmholtz resonator consists of an air cavity whose dimension is small in comparison with the wave-length of the sound to be detected. One type of resonator A is a pear shaped vessel with a pip at the back and an aperture called neck at the front. The other type B is a cylindrical vessel with a rather long neck and its volume may be continuously variable.

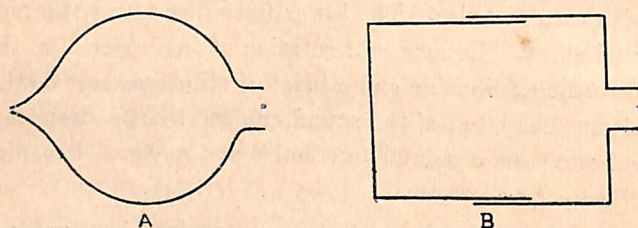


Fig. 45

To calculate the natural frequency of vibration of a resonator of volume v and a neck of length l and radius r , we must suppose that the air in the neck acts as a piston and rarefaction or condensation at any instant is uniform in the cavity.

Let ξ = displacement of the "piston" of sectional area s at any instant t and let δp be the increase in pressure in the cavity. Then

$$s\delta p = s\rho l \frac{d^2\xi}{dt^2}, \text{ where } \rho = \text{density of air.}$$

Since pressure change in the cavity is adiabatic, $\delta(pv^\gamma) = 0$

$$\text{whence } \delta p = -\gamma p \cdot \frac{\delta v}{v}$$

$$\therefore \rho l \frac{d^2\xi}{dt^2} = -\gamma p \cdot \frac{\delta v}{v} = -\gamma p \cdot \frac{s\xi}{v}$$

$$\text{Or, } \frac{d^2\xi}{dt^2} + \frac{\gamma ps}{\rho \cdot v} \xi = 0$$

whence frequency of vibration is

$$N = \frac{1}{2\pi} \sqrt{\frac{\gamma ps}{\rho lv}} = \frac{c}{2\pi} \sqrt{\frac{s}{lv}} \quad \left(\because c^2 = \frac{\gamma p}{\rho} \right)$$

$\frac{s}{l}$ is known as the conductivity of the vessel and has the dimension of a length. For a circular aperture $l \rightarrow 0$; Rayleigh shows that in this case conductivity is equal to the diameter of the aperture. Helmholtz and Sondhauss experimentally obtained that natural frequency as calculated is higher than the observed frequency. This is due to open end effect and l should be corrected to $l + 0.6r$.

The expression for frequency as deduced, refers only to the fundamental tone of the resonator, but a large number of tones are possible.

2. (a) Distinction between a pipe and a Helmholtz resonator:

(1) In a Helmholtz resonator, the state of rarefaction or condensation is the same throughout the cavity at any instant. In a pipe resonator the minimum length of the pipe must be at least a quarter of the wave-length of the tone to be detected and state of condensation varies from point to point.

(2) Damping in a Helmholtz resonator is very small. If the volume of the resonator is large and area of the neck is small, then the vibration in a tuned resonator will persist for a long time. Since the damping is small, a Helmholtz resonator is highly selective and the response is very sharp.

2. (b) Detection of a frequency by Helmholtz resonator:

A series of resonators of different natural frequencies may be used. The ear is placed at the nipple of the resonator which speaks when the frequency of a tone sounded near the neck of the resonator equals its natural frequency. A thin reed of mica fixed across the aperture of the resonator and tuned to its natural frequency is a very sensitive arrangement of detecting the frequency of a tone. At resonance, there will be a vigorous vibration of the reed which can be detected by a ray of light reflected from it.

To analyse a note consisting of several harmonics, a large number of resonators of fixed volume will be required. Hence it is very helpful to use a resonator of continuously variable volume; this may be achieved by using a sliding piston or a water column of variable height.

There are several objective methods of detecting resonance in the resonator. A hot wire microphone placed across the collar or a Rayleigh disc placed at 45° to the axis of the neck is a very sensitive arrangement to detect the resonant condition.

For greater sensitivity a **double resonator** may be used. This consists of two resonators of same natural frequency but of different conductivities. The volume of the outer resonator is generally greater than that of the inner one. A Rayleigh disc or a hot wire microphone is placed in the neck of the inner resonator. The arrangement is extremely sensitive as well as selective and hardly perceptible sound can be easily detected by it.

Measurement of intensity : Rayleigh disc :

Rayleigh observed that a light disc suspended at an angle to the axis of a sounding cylindrical resonator tended to turn at right angles to the axis.

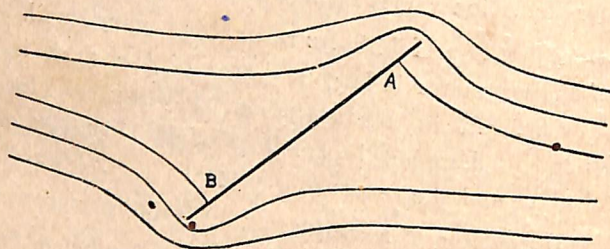


Fig. 46

Let us suppose that a light disc is suspended in a stream of air, so that it is capable of rotating freely about a vertical axis. If we study the nature of the stream lines, we shall see that these are completely stopped at points A and B, and hence these must be the regions of maximum pressure. It is clear from the figure that the lines of flow will be unaltered for a reversed direction of flow. Thus a couple will act on the disc

to turn it across the stream. In the case of a circular disc of radius a and the normal to the plane of the disc making an angle θ with the direction of the undisturbed stream the moment of the couple acting on it is, as developed by König

$$M = \frac{2}{3} \rho a^3 V^2 \sin 2\theta$$

where ρ = density of the fluid

and V = velocity of the stream.

Thus if $\theta = 45^\circ$, the moment of the couple is maximum.

In the case of an alternating current of air, the mean value of the square of the velocity of the stream is to be taken.

In practice the disc is of mica of radius 1 cm. and suspended by a thin quartz fibre 10 cm. long. The disc is suspended at an angle of 45° to the plane of the opening of a Helmholtz resonator; a ray of light reflected by the disc falls on a scale placed outside.

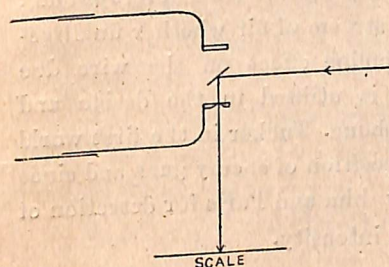


Fig. 47

Since the turning couple is proportional to average V^2 , it is also proportional to intensity. For a small angle of deflection, the intensity is directly proportional to it; the magnitude of the torque can, however, be measured by a torsion head by bring-

ing back the disc to its original position.

A Rayleigh disc can be used with a tube resonator or a double resonator. In the latter case it is placed in the neck of the inner resonator and selectivity of the arrangement can be made very high when the natural frequencies of the two separate resonators forming a double resonator are same. With a resonator of a continuously variable volume, Rayleigh disc is a very useful instrument of measuring intensities at different frequencies.

Corrections in König's formula : König's simple theory of a Rayleigh disc ignores the following factors.

(1) The diffraction of sound radiation by the disc : This will be negligible when the diameter of the disc is small compared with the wave-length of the sound.

(2) Viscosity effect of the medium : The motion of the disc due to motion of the medium in addition to its small rotation : This effect cannot be neglected when intensity of sound is measured in a heavy medium such as a liquid. In this case the moment of the disc must be multiplied by a factor $(1-\beta)^2$ where β = ratio of the amplitude of velocity of the disc to that of the medium. The moment M then is

$$M = \frac{4}{3} \rho a^3 V_m^2 (1-\beta)^2 \sin 2\theta$$

when the disc is set at an angle θ , V_m being the velocity of the particles constituting the medium.

(3) Possible transverse vibrations of the disc : If the vibration of the medium coincides with one of the modes of transverse vibrating of the disc, then the disc becomes useless for the intensity measurement.

3. (a) Measurement of intensity by hot wire microphone :

If a heated wire is placed in a stream of air whether unidirectional or alternating, there is a cooling effect on the wire due to air draught. This principle is utilised in the design and construction of a hot wire microphone. Tucker in the first world war designed it to locate the position of enemy guns and since then it has been developed by him and Paris for detection of sound and measurement of sound intensity.

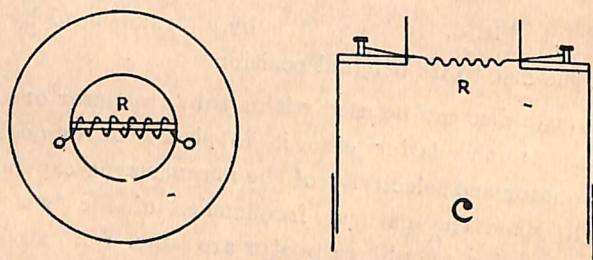


Fig. 48

R is a platinum wire made from Wollaston wire with a silver sheath which is removed by nitric acid after it has been wound on a rod of glass enamel and mounted on a mica disc with a central hole. The two ends of the wire are connected to two annular rings of silver mounted on opposite sides of the mica disc. The platinum wire is very thin (0.0006 cm. diameter) and

has a resistance of about 140 ohms at 10°C . The temperature rises to just below dull red heat (400°C) when a current of about 30 milliamperes passes through it and the resistance then is about 360 ohms. The arrangement is mounted in the neck of a Helmholtz resonator C . When the resonator responds to a tone of suitable frequency, air draught in the neck cools the platinum grid causing the resistance to fall.

The decrease in the resistance consists of three components δR_1 , δR_2 and δR_3 . The change δR_1 is the steady component proportional to u^2 , whereas δR_2 is oscillatory and proportional to $u \sin pt$ and δR_3 to $u^2 \cos 2pt$ when $p = 2\pi N$, N being the frequency of the tone and u the maximum velocity of the air-stream in any direction. δR_3 is very small and is usually neglected.

The steady drop δR_1 which is proportional to the intensity is measured by Wheatstone bridge method. It is made one arm of the bridge, the heating current of 30 milliamperes being supplied by the battery of the bridge. The out of balance current in the galvanometer, when the grid is cooled at resonance,

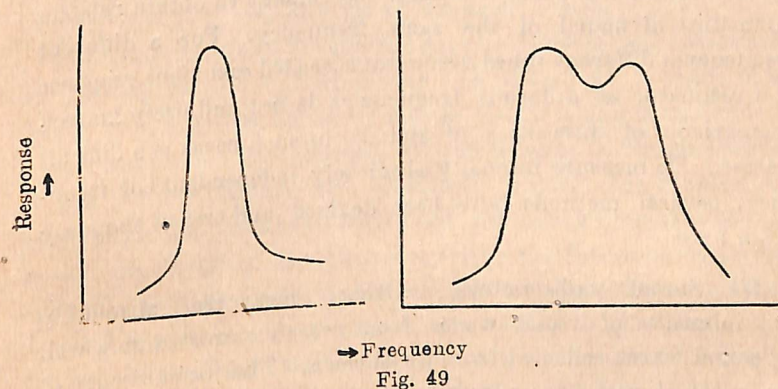


Fig. 49

gives a measure of the intensity of the sound. That this is so has been experimentally verified by Tucker and Paris who showed that the out of balance current is inversely proportional to the square of the distance between the resonator and an electrically maintained turning fork in open air, the resonator being tuned to the frequency of the fork.

To measure δR_z , the oscillatory potential difference across the wire is amplified by a valve amplifier and the output current which is proportional to the amplitude of vibration is passed through a suitable measuring device.

Paris used the hot wire microphone with a double resonator. The vibrations in this case are coupled and there is a response over a range of frequencies. The grid can be calibrated over this range by a siren of known constant output. If the frequencies of the separate resonators constituting the double resonator are made equal, then there is almost uniform response over a small width of frequencies, the response curve showing two peaks with a small central dip. Such a double resonator is very sensitive and selective, specially so when the frequency is low. The nature of response with a single and double resonators and with a hot wire microphone is shown in figure 49, response being measured by the out of balance current.

3. (b) Absolute measurement of intensity :

A Rayleigh disc with a resonator or a hot wire microphone with a resonator is a very suitable instrument to obtain relative intensities of sound of the same frequency. For a different frequency a different tuned resonator is needed and since response of a resonator at different frequencies is not definitely known, comparison of intensities of sounds in such cases is a difficult matter. To measure intensity absolutely independent of frequency, several methods have been devised and two of them are given here.

(i) **Sound radiometers :** Altberg measured absolutely the intensity of sound waves from pressure exerted on a wall by sound waves reflected from its surface. The basic theory of the method can be understood from the following treatment due to Larmor.

Let us suppose plane sound waves of velocity c are incident perpendicularly on a totally reflecting plane wall moving parallel to itself with a velocity u in a direction opposite to that of the waves. Let E be the energy density of the incident waves. In one second, a length of wave train $c+u$ is incident on the wall

and hence, energy of amount $(c+u)E$ is compressed in a length $c-u$. If $E+\delta E$ is the energy density in the reflected wave,

$$(c+u)E = (c-u)(E+\delta E)$$

$$\text{or, } \frac{E+\delta E}{E} = \frac{c+u}{c-u} = \left(1 + \frac{u}{c}\right) \left(1 - \frac{u}{c}\right)^{-1} \\ = 1 + \frac{2u}{c} \text{ approximately}$$

when $c \gg u$

$$\text{Thus } \frac{\delta E}{E} = \frac{2u}{c} \quad \text{or, } c\delta E = 2uE$$

If P is the pressure on the wall, work done per second in moving unit area of the wall against this pressure accounts for the increase in the energy in a length c in front of it,

$$\text{or, } P \cdot u = c\delta E = 2uE$$

$$\therefore P = 2E$$

Thus radiation pressure is the total energy density in front of the wall. If the wall does not reflect at all $P = E$.

The steady pressure exerted by sound waves was measured by Altberg in the following way. A little hole is made in a wall. This is nearly closed by a loose piston which is suspended from one arm of a very delicate torsion balance. If the constants of the torsion balance are previously known, the couple and from that the pressure exerted on the piston on which sound waves are incident can be measured by the deflection of a spot of light. To have a measurable deflection very intense sources of sound were taken, such as glass rods excited longitudinally.

In an experiment, Altberg calculated the pressure as 24 dyne per square centimeter from which energy density becomes 12 erg. per c.c. (supposing waves to be completely reflected). By multiplying the energy density by the velocity of sound (about 35000, cms/sec. approximately) the intensity comes to be about 4200 ergs per sq. cm. per second.

(ii) **Optical method :** Let us suppose interference fringes are obtained from two beams of light. Let one of them pass across a space A through which the sound waves travel where as the other passes through undisturbed air. When sound

waves are allowed to move through A , the width of a fringe will appear to increase. From the change in the width, the density variation and hence the intensity of the sound waves through space A may be calculated. The increase in the width of a fringe may be explained by the fact that rapid variation of path difference between the two beams occurs as a result of density change of the medium in space A . Hence the position of a fringe in the field of view rapidly changes about the mean position and due to persistence of vision the width seems to be larger. The amplitude of motion of a band may be found out by a stroboscope.

The instrument can be calibrated by taking air at different pressure in the space A , and noticing the shifts of the fringe system.

The interferometer method was introduced by Topler and Boltzmann and developed by Raps.

4. **Intensity and loudness:** Physically the intensity of a sound wave is the energy passing per second across unit area perpendicular to the direction of propagation and in the case of a plane wave we have seen that intensity

$$W = \frac{2\pi^2 a^2 \rho_0 c^3}{\lambda^2} = \frac{1}{2} \left[\frac{\delta \rho_{max}^2}{\rho_0 c} \right]$$

[See pages 70 & 71, Chapter VI]

The subjective sensation of loudness is, however, not proportional to intensity. The range of audible frequency is from about 20 cycles per second to about 20,000 cycles per second and the sensitivity of normal ear is different at different frequencies. The maximum pressure variation ($\delta \rho_{max}$) to produce a just audible sensation of sound at 25 cycles per second, is as large as 10 dynes/cm², while it is as low as '00076 dynes/cm² at 1000 cycles per second. The ear is most sensitive to a sound of about 3500 cycles per second when the pressure at threshold of audibility is '00008 dynes/cm² only; the displacement of an air particle at this pressure is smaller than the diameter of a hydrogen molecule.

The subjective sensation of loudness at a frequency is given by what is known as Weber-Fechner Law. The increase δs in

the sensation of loudness is proportional to $\frac{\delta I}{I}$, where δI is the small increase of intensity I . Hence, increase in the sensation level of the sound is $s_2 - s_1 = k \cdot \log_{10} \frac{I_2}{I_1}$, where k is a constant.

If $k=1$ then sensation level of two sounds is given in Bels and is thus equal to $\log_{10} \frac{I_2}{I_1}$ bels.

The intensity level of two sounds is given by $\log_{10} \frac{I_2}{I_1}$ and is expressed in bels. If expressed in decibels, the intensity level is $10 \log_{10} \frac{I_2}{I_1}$ decibels, since 1 bel = 10 decibels. The reference intensity for loudness comparison is 10^{-16} watt/cm² at 1000 cycles per second (at r.m.s. sound pressure = '0002 dynes/cm²) which is the threshold of audibility at that frequency. A sound having an intensity 10 times this value is said to have an intensity of 1 bel or 10 decibels.

The sensation level of a sound of any frequency is measured by the ratio of its intensity to the threshold intensity at that particular frequency. Or, sensation level = $10 \log \frac{I_1}{I_0}$, decibels, where I_1 = intensity at that frequency and I_0 = threshold intensity at that frequency. Since I_0 is very large at low or high frequencies, the sensation level for a particular value of I_1 is small at high or low frequencies.

4(a) **Measurement of equivalent loudness:** Loudness of a sound is measured relative to the loudness of the standard reference pure tone of frequency 1000 per second and intensity 10^{-16} watt/cm². This reference tone may be produced free from harmonics by a valve oscillator. The sound whose loudness level is to be found out and the standard reference tone is alternately heard by the observer from the same distance. The pure reference tone is gradually increased in intensity till the two sounds appear to be equally loud. The increase in intensity in decibels of the reference tone from that at threshold level is the measure in phons of the equivalent loudness of the

sound. The intensity level, sensation level and equivalent loudness in phons are numerically equal at 1000 c. p. s, the frequency of the standard reference tone.

5. Measurement of quality : The sound produced by a musical instrument reaches our ear through air. Hence if the vibrations of the free air particles could be recorded, the quality of the note emitted could be found out by performing Fourier analysis of the vibration curve. The essential requisite of a faithful recorder of vibrations is that it must have a uniform response over the frequencies comprising the audible range.

(a) **Miller's Phonodeik :**

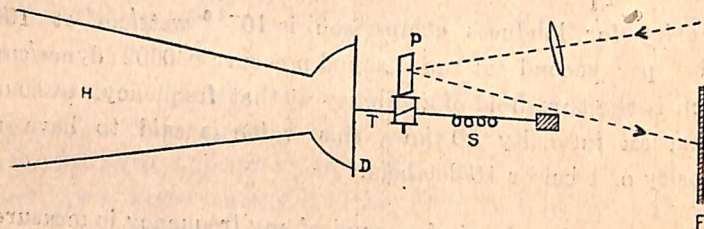


Fig. 50

H is a conical horn at the end of which on soft rubber rings there is a diaphragm of glass of thickness about 0.003 inch. A silk fibre or a platinum wire (0.0005 inch thick) attached at the centre of the diaphragm passes round a small pulley fixed with a small steel spindle mounted on jewelled bearings and is then connected to a spring. On the spindle there is a small oscillograph mirror which rotates when the diaphragm moves. A ray of light is reflected from the mirror to a moving film *F*. The motion of the diaphragm is magnified about 2,500 times.

The serious defect in the above instrument is that the response is not uniform; the natural frequency of the diaphragm may be made very high, but it is impossible to eliminate resonance effect due to the horn and the air column. Miller calibrated his phonodeik by tones of different frequencies, but of constant loudness and could incorporate correction terms in the evaluation of the Fourier coefficients obtained from the analysis of the trace of the curve obtained on the sensitised film.

(b) **Cathode ray oscilloscope :**

The Cathode ray oscilloscope is the ideal instrument for analysis of frequency and studying quality of a musical note.

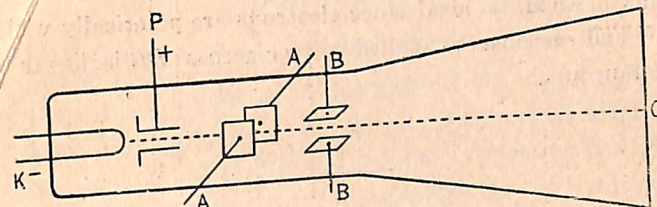


Fig. 51

The instrument is an evacuated tube containing a cathode *K* and an anode *P*. The cathode *K* is a coated filament emitting electrons when heated by an electric current. When the potential of *P* is made positive with respect to the cathode by application of a potential difference of a few hundred volts, electrons emitted by the cathode are accelerated towards the anode *P* and through a hole in the anode emerge out as a thin beam and strike at *O* the centre of a fluorescent screen at the end of the tube and cause fluorescence there. *A, A* and *B, B* are two pairs of deflecting plates and any potential difference established between *A, A* or *B, B* will cause the electrons to be deflected horizontally or vertically.

The sound is received by a distortionless microphone or a calibrated microphone whose response to different frequencies is known.

The fluctuating current on the microphone is amplified by a valve amplifier and the amplified voltage is applied across the plates *B, B*. Due to persistence of vision a vertical line through *O* will appear on the fluorescent screen. The displacement curve of the electron beam may be obtained on a photographic film when it is moved at a constant rate across the line in the horizontal direction. To obtain a stationary trace of the curve, the electrons are swept horizontally at a constant frequency by applying a

fluctuating voltage between A, A of frequency equal to that of the fundamental of the note to be studied and wave form of sawtooth type. The curve then can be photographed and studied and all the Fourier coefficients of the waveform calculated.

This method is ideal since electrons are practically without inertia and response to applied voltage across BB is linear and instantaneous.

CHAPTER XIV

VORTEX SOUNDS AND MAINTAINED VIBRATIONS

1. Vortex Sound : (a) Aeolian tones :

Let us consider a fluid streaming past a solid, *e.g.*, a cylinder dipped in the fluid as shown in the figure. In the region behind the cylinder at B and C the fluid will have motion as indicated by the arrows and form vortices rotating in the opposite directions if the velocity of the fluid is above a certain minimum value. If the velocity reaches a sufficient value, these vortices attaining a sufficient size will be carried along the stream, the detachment taking place alternately from each side.

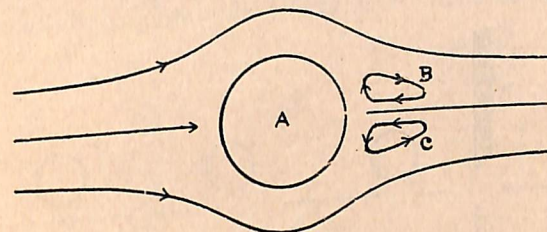


Fig. 52

The formation of these double series of vortices is due to the viscous drag between the fluid and the surface of the obstacle. If the distance between two rows of vortices be h and that between the successive vortices in the same row be l , then it has been experimentally shown that $\frac{h}{l} = .28$. Thus the distance between two successive vortices in the same row will be independent of the velocity ; with the increase of the velocity of the stream, the rate of formation and detachment of the vortices will be increased.

Let us consider a string of diameter d under tension placed in the path of a wind. Vortices will be alternately detached from the sides in the rear of the string and if the velocity is constant, there will be periodic cross-forces on the string as the

two types of eddies are formed and detached. It has been found experimentally that the wave-length λ in the stationary air due to alternating pressure wave is $\lambda = 5.4d$, where d = thickness of the wire. Hence the frequency of the note due to cross-forces of the eddies detached is $N = \frac{v}{5.4d}$. If the frequency of the periodic disturbance is equal to that of one of the harmonics of the string under tension, then the string will vibrate in resonance. Such a tone is known as *Æolian tone*. If several wires with different diameters are fixed up in a frame and tensions adjusted such that the pitches of the fundamentals of all the strings are the same, then wind passing through them may excite one or more wires and produce an appropriate overtone. Such an instrument is known as a *Æolian harp*.

(b) **Jet tones**: Let us take the case of a fluid issuing out in a stream through a rectangular slit into an infinite ocean of

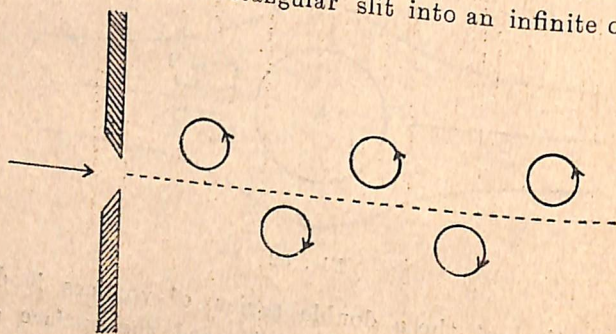


Fig. 53

the same fluid. Due to the surfaces of discontinuity formed, two rows of vortices will be produced, the upper ones revolving in the anticlockwise direction and the lower ones in the clockwise direction as shown in the figure. If h is the distance between the two rows of vortices and λ the distance between two successive ones in the same row, then the relation between them is given by Karman formula $\frac{h}{\lambda} = 28$.

The frequency of the jet tone formed due to pressure fluctuations in the stationary fluid is given by $N = 0.55 \frac{v}{a}$,

where v = velocity of efflux of the fluid and a the width of the slit.

The tones produced by a jet are weak and uncertain. This is due to the fact that there is no solid resonator here to resonate to the tone produced.

(c) **Edge tones**? Masson and Sondhauss first observed that when a blast of air is directed against a sharp edge, tones can be produced.

Let us suppose a blade shaped stream of air issues from a slit and impinges on a sharp edge as shown in the figure. Two vortex rows are formed, the vortices in the two rows revolving in opposite directions. In the simplest case, a vortex revolving in the anticlockwise direction just reaches the tip of the wedge

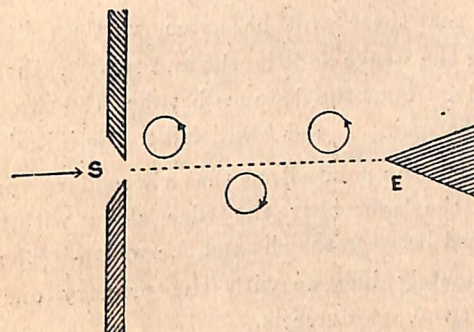


Fig. 54

when the next in the same row is just near the slit. In this position there is one vortex rotating in the clockwise direction at the middle position of the two vortices in the upper row. Experiments have shown that for a velocity of the efflux U there is a minimum distance $x = x_0$ between the slit and the wedge when such vortices can be formed.

If V is the velocity with which the vortices move towards the edge, then the pitch of the edge tone is given by $N = \frac{V}{\lambda}$ when x is the distance between the slit and the tip of the wedge; λ the distance between two vortices in the same row is the wave-length of the disturbance. If the velocity of efflux of the air stream is U , then $V = KU$ when K is a constant. Hence frequency of the

edge tone is $N = \frac{KU}{x}$. Thus if U is increased keeping the wedge slit distance constant, the frequency also increases till vortices suddenly re-arrange with a jump in frequency. Let us suppose U is kept constant and the wedge is gradually shifted away from the initial position corresponding to $x = x_0$, the minimum distance for formation of an edge tone for the velocity of efflux U . With the increase of wave-length x , the frequency will decrease. If h is the distance between the two vortex rows, then according to Karman's formula $\frac{h}{x}$ will be always constant; hence h will increase with increase of x . If x is further increased, then at a value of $x = 2x_0$, the frequency will suddenly jump to an octave and will be equal to N_0 corresponding to x_0 . This is explained by the fact that there will be three vortices in the upper row between the wedge and the slit and two in the lower one simultaneously. Thus the distance between the two consecutive vortices will be again x_0 , and h will fall to the initial value.

E. G. Richardson pointed out that a secondary vortex system is formed on the layer near the edge along with the primary vortices formed between the slit and the wedge. The secondary vortices ultimately line up with the primary ones with the formation of two vortex streets.

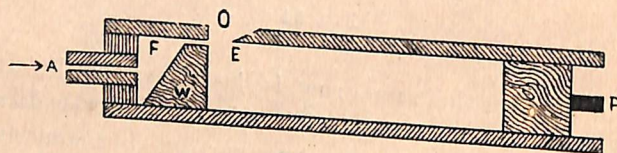


Fig. 55

2. Organ pipes:

(a) Flue organ pipe:

The vibrations of air in a flue organ pipe are maintained by edge tones formed when air rushes through a slit and impinges on an edge.

A flue organ pipe may be either open or closed. In the closed type, P is an adjustable stop closing one end of the tube. These pipes have square cross-sections or circular cross-sections when made of wood or metal. The wind from a windchest

supplied by foot bellows at a constant pressure of several inches of water enters the mouth piece through the narrow end A and passing the narrow slit F (flue) strikes the edges E which is a bevelled edge of the pipe. The speaking length of the pipe from E to P can be adjusted by the stop P in a closed pipe.

It was Wachsmuth who first discovered that the vibrations in the pipe are coupled to the edge tones. We have seen earlier in this chapter that the frequency of the edge tone is determined by the distance x which is equal to OE , the opening of the pipe and the velocity of efflux of the air stream. The organ pipe is so designed that at a normal blowing pressure, the pitch of the edge tone equals that of the fundamental of the pipe. Hence for a pipe meant to produce a tone in the bass of the scale OE is made comparatively large.

(i) Effect of overblowing and underblowing the pipe:

Let us suppose that the blowing pressure be gradually increased from the normal blowing pressure P_0 corresponding to the production of the fundamental vibration in the pipe. In the absence of the pipe the frequency of the edge tone will be given by the relation $\frac{V}{nx} = \text{constant}$, where $x = OE$ or submultiple of OE and V the velocity of efflux which increases with blowing pressure, and n frequency of the tone. But things will be different when edge tones and vibrations in the pipe are coupled. The latter are less damped and their period is not alterable to any appreciable extent. Thus the period of the pipe will be forced on that of the edge tone. The distance between consecutive vortices in the same row will be altered to such an extent that the natural vibrations of the pipe are maintained. This accommodation will continue till the pressure is P' when V' the velocity of efflux of the airstream is such that the pitch of the free edge tone equals the pitch of the next harmonic of the pipe; then the vibrations of the pipe will jump to that pitch.

Similarly, if the blowing pressure is reduced from the normal blowing pressure P_0 , the pitch of the edge tone natural to it in the absence of the pipe, will be lowered; but here also the period of the pipe will be forced on it. Very soon the blowing

pressure is reduced to such an extent that edge tone can hardly be produced. Still the pipe tries to impose one of its overtones upon the vortex production and the fundamental is sounded very feebly. Thus if V be the velocity of efflux when edge tone is in resonance with the tube producing its fundamental, it is still possible to have the fundamental sounded at velocity $\frac{V}{N}$ when the spacing between two consecutive vortices of the same type or the wave-length is $\frac{x}{N}$.

(ii) **Effect of the width of the pipe :** The end correction at the blown end is very large, about twice or thrice the radius of the pipe. This end correction is also dependent on frequency ; hence the vibrations in the pipe do not form a strictly harmonic series. For a narrow tube where the end correction is small, the natural vibrations depart less from a harmonic ratio.

(b) **Reed pipe :** The vibration of an air column can also be maintained by a reed. In this type of organ pipes a reed of brass is at the vertex of the pipe proper which is conical. The reed is attached in the side of the opening of a small tube called a shallot covering a hole in it. The end of the conical tube just fits into the shallot which is fed with air from a chamber called boot. The period of the fundamental vibration of the reed can be adjusted with the help of a wire spring.

Due to rush of air into the shallot, the reed is deflected and again springs back due to tension. The overtones of the reed are not harmonic ; hence the resultant note is the fundamental of the reed which is also the fundamental of the pipe.

3. **Singing flame :** Higgins in 1771 observed that a jet of hydrogen burning in an open tube sometimes emits a musical note. De la Rive tried to explain that the note was due to periodic condensation of water vapour formed by combustion of hydrogen. But Faraday showed that a note could be produced burning carbon monoxide in air, in which case there is no formation of water. His idea was that the note was due to successive explosions of the combustible gas. Sondhauss first showed that the note was due to intermittent heating of air near the jet.

and the pitch was related to the length of the tube. A satisfactory explanation of the singing flame was given by Lord Rayleigh.

The following arrangement may be made to obtain a singing flame.

An upright supply tube 10 cm. high with a pinhole burner at the top is fastened with a horizontal gas pipe. The jet is lighted into the wider glass tube of about 1.5 cm. in diameter and 30 cm. in length and supported by a clamp. When the jet is at about 7 cm. or 8 cm. from the lower end of the tube, the flame may begin to sing spontaneously. The flame may be coaxed to sing by gradually decreasing the size of the flame by lowering gas supply. The pitch of the note is equal to $\frac{c}{2l}$, where c is the

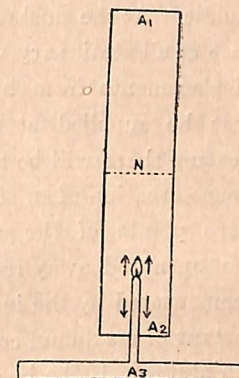


Fig. 56

velocity of sound and l the length of the tube which shows that vibrations are stationary. When the jet burns steadily without producing any note, it appears like a band in a rotating mirror ; when the flame sings, it has a tooth like structure in the rotating mirror as shown in the figure, which shows that the flame burns intermittently with a period equal to that of the note.



Fig. 57

Lord Rayleigh showed that the most important factor was the phase of heat supply by the jet relative to that of the vibration in the tube. The vibrations are most encouraged when heat is supplied at the moment of greatest condensation and near the region of greatest condensation.

Since the vibrations in the tube are stationary, the displacements of particles between a node and an antinode are maximum at maximum condensation. Hence, heat is supplied at the moment of maximum displacement. This differs from the case

of ordinary forced vibration which is most encouraged where the impressed harmonic force is maximum at displacement equal to zero. The difference is due to the intermittent instead of harmonic nature of the heat supply.

Let us suppose stationary waves are formed in the pipe with one of the antinodes A_1 at the top of the tube and a node at the middle. At the instant of maximum condensation, as in the case of all stationary waves, the air particles have the maximum displacements from their zero positions towards the node. Let heat be supplied at that instant to air at and above N . Due to heating there will be an increase in the pressure amplitude, and hence creation of an additional restoring force on the air. Thus zero points of the particles will be shifted in the direction of the open end away from N . The losses due to friction etc., are compensated by the increase in the pressure amplitude at the instant of maximum condensation, and hence vibrations will be maintained. If the heat supply is made at the instant of greatest rarefaction when the motion of the particles is away from node, then due to the sudden application of the pressure the vibration will be discouraged. The two cases are shown by fig. 58.

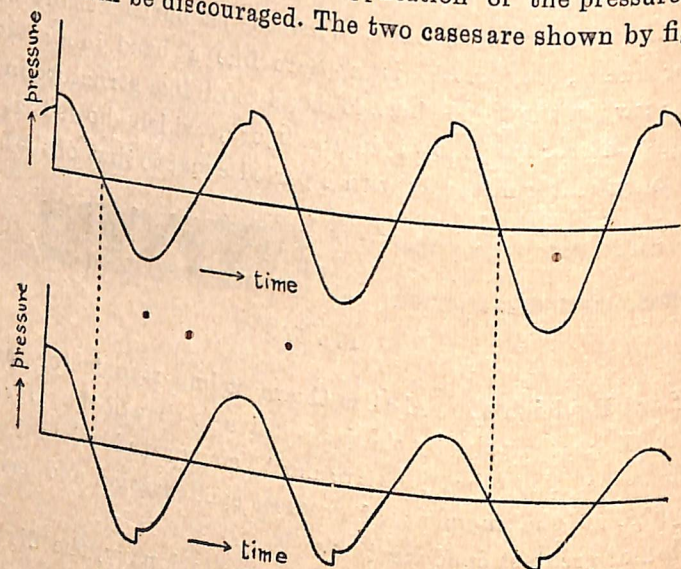


Fig. 58

In the two cases discussed above there will be no change in the time period as amplitude of vibration has no effect on time.

period. In the case of heat supply being made at zero pressure when rarefaction is just beginning, a sudden applied pulse of pressure will delay the displacement in being zero and hence time period will be increased; but there will be neither any encouragement nor any discouragement to the vibration. By a similar reasoning sudden heating at zero displacement when condensation is just starting, will decrease the time period. This also will neither encourage nor discourage maintenance of vibration.

The most important thing is that the oscillation of the flame maintains the correct phase relationship with the vibration in the tube. This is possible because the stationary waves are also formed in the supply tube with a node above the jet and an antinode at where the supply tube connects the longer gas tube. At condensation the particles in the supply tube and the singing tube rush towards the node. At rarefaction the motion is always from the node and thus heat supply to the air in the tube is minimum as the flame is partially withdrawn. The node of the supply tube must be above the jet; otherwise there could be no motion of combustible gas particles into the outside tube through a node. In the figure N, A_1 correspond to nodes and antinodes of the supply tube and N, A_s are the same of the singing tube.

To sum up, the heat supply must be at the moment of greatest condensation and the jet should be near the node of the singing tube. The length of the tube should be a little more than $(n+1)\frac{\lambda}{4}$ where n is an integer.

4. Trevelyan rocker: It is an example of vibration maintained by heat. The rocker consists of a prism of brass or copper with a rod terminating in a knob. The prism has one of the edges removed and a groove with two ridges are made as shown in the figure.

A block of lead with a rounded top is taken and its surface is cleaned by scraping. The rocker is heated to a temperature below the melting point of lead and then placed on the lead block. By application of a smart tap near the ridge, vibrations may be started in the block.

The explanation of the vibration may be given in the following manner. Let us suppose that during vibration, one ridge is in contact with the block, whereas the other ridge is at a raised

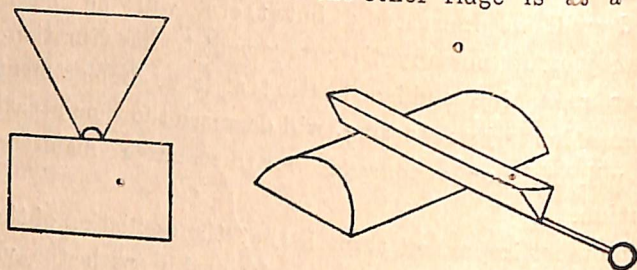


Fig. 59

position. Conduction of heat takes place through the ridge in contact with the block and a hump is created due to expansion for local heating. The rocker then is tilted in the other direction, this time the other ridge being in contact with the block and the previous one raised to some height. In this way the vibrations are maintained by the heat flow from the rocker to the block. The periodic tilting forces the table on which the rocker is placed to vibrate and a note is produced. In order that the vibrations are maintained, there must be a lag in the contact between the ridge and the block and the creation of a hump there due to heating.

CHAPTER XV

MUSICAL SCALE : CONSONANCE
AND DISSONANCE

1. Consonance and dissonance : Two musical notes sounded together may produce a sensation which may be either pleasant or disagreeable to the ear. The notes produce consonance or concord when the resultant sound is pleasant, and dissonance or discord when it is irritating. When the ratio of the two frequencies of the individual notes can be expressed in small whole numbers, the resultant effect will be generally concordant.

According to Helmholtz, the discord is due to formation of unpleasant beats by the component notes or some of their overtones. This is similar to the unpleasant sensation in the eye when a light flickers. The unpleasantness is negligible when the flicker is either too slow or too rapid. In the same way if the number of beats formed per second by the component notes is less than four, there is practically no discord. But the upper limit of the number of beats per second when discord vanishes depends on the pitches of the component notes. Thus 33 beats per second produce the maximum dissonance at a frequency 512, while there is no dissonance when the number of beats is above 78 per second at this frequency. The discord between the component notes depends in a compound manner on the ratio of the frequencies of the two notes as well as the difference in the frequencies. Since discord may be due to beats between the fundamentals of the two notes or any of the harmonics, the dissonance is dependent on the quality of the component notes.

2. Musical interval : In music, the absolute values of frequencies of two notes are less important than the ratio between the two. The musical interval between two notes is the ratio of their frequencies. The compatibility or agreeability of the two notes depends on a simple ratio of the frequencies. The

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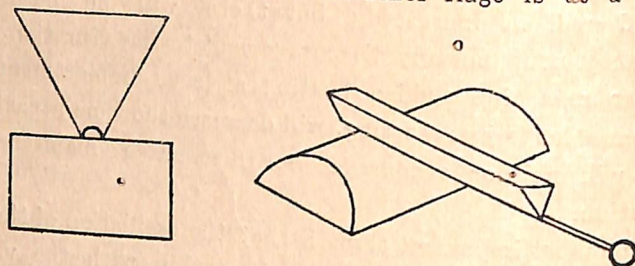


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musical intervals are expressed by the several terms as given below.

Interval	Ratio of frequencies	Intervals	Ratio of frequencies
Unison	1 : 1	Major Third	5 : 4
Octave	2 : 1	Minor Third	6 : 5
Fifth	3 : 2	Major Tone	9 : 8
Fourth	4 : 3	Minor Tone	10 : 9
		Semi Tone	16 : 15

Harmony: Notes that are pleasant in combination have simple frequency ratios. When the notes of simple frequency ratios are sounded together, harmony is produced. Three notes of frequency ratio 4 : 5 : 6 constitute a triad, while in the common chord there are four notes of frequency ratio 4 : 5 : 6 : 8. Concord is an essential part of harmony. The melody is the total effect of a sequence of notes in the scale played one after another.

Diatonic scale: A musical scale is a scale of frequencies which are divided into successive groups, each group being limited by the octave of the gravest tone of the group. The construction of a musical scale depends on how each group is to be divided into notes. The musical scale extends over 7 octaves from range 32 to 4000; generally three octaves suffice.

In the diatonic scale, there are eight notes from the lowest of the series called the tonic or key note to its octave. The notes as named in western countries are Do, Re, Mi, Fa, Sol, La, Si, do from the key note to the octave. In India the notes are named Sa, Re, Ga, Ma, Pa, Dha, Ni, Sa. Helmholtz first denoted the notes by the letters C, D, E, F, G, A, B, c.

The frequency relation, etc. of the notes comprising the diatonic scale are given in the following table.

Symbol	Notes	Frequency relation with Tonic	Musical interval between successive notes
C	Do	1	
D	Re	9 : 8	9 : 8
E	Mi	5 : 4	10 : 9
F	Fa	4 : 3	16 : 15
G	Sol	3 : 2	9 : 8
A	La	5 : 3	10 : 9
B	Si	15 : 8	9 : 8
c	do	2 : 1	16 : 5

The third column gives the ratio of the frequency of the note with respect to that in the tonic or key note C. The fourth column gives the musical interval between the successive notes. Thus we see there are three major tones expressed by the ratio 9 : 8 and two minor tones expressed by the ratio 10 : 9 in the scale.

In Indian music, the major tones, minor tones and the semi-tone are represented by 4, 3 and 2 *shruties* respectively. Thus in the octave there are altogether 22 *shruties*.

The ratios of the notes in the diatonic scale are the ideal ones and there will be no dissonant harmonics if suitable combinations of notes are taken. Thus the scale is the ideal one to produce the best harmonious combinations and no scale can be designed to replace it. The great defect of the scale is its limited power of modulation. The characteristic of modern western music is that the key note is frequently changed. Let us suppose, the key note at one time is C of frequency 256; if the tonic is changed to E of frequency 320, new notes are

required when same intervals are to be used in both cases. Thus with a diatonic scale, change of key note is not possible.

Equitempered scale : In the equitempered scale there are twelve notes in the octave and the musical interval between any pair of successive notes is the same. Thus if x is the interval called semitone between two successive notes, then $x^{12} = 2$, whence $x = 1.05946$.

The intervals between the successive notes of the diatonic and the equitempered scale are given below.

Scale	C	D	E	F	G	A	B	c
Diatonic	1	1.125	1.250	1.333	1.500	1.667	1.875	2
Equi-tempered	1	1.122	1.260	1.335	1.498	1.680	1.883	2

The equitempered scale is essential for an instrument with keys like piano, organ etc., or stringed instruments with frets.

Musical pitches : The pitch of note C is 256 as used in the laboratory. Early in the present century there were three main pitches in use. (1) Old Philharmonic with A at 442 c. p. s. (2) New Philharmonic with A at 439 c. p. s. and (3) French Diapason Normal with A at 435 per second. In 1939, an international committee recommended that A should be taken as 440 c. p. s. All the other notes can be calculated on this basis.

Vowel sounds : A vowel sound has a characteristic quality of its own and it cannot be produced by any musical instrument. The wind from the lungs sets the vocal cords which consist of two membranous reeds under tension to vibrate. The vibrations travelling up the mouth produce a vowel sound determined by the shape of the mouth. The consonants are produced by the sudden stoppage of the lips or sudden movement of the tongue.

According to Helmholtz, a vowel sound is due to the harmonic components of the cord tone reinforced by resonance in the oral cavity consisting of the mouth, nose and throat. The oral cavity as a whole may give two simultaneous resonances to a

vowel sound. The tongue divides the mouth into two cavities, the larger one having a resonance between frequencies 300 to 850 and the smaller one between 600 to 2,500. D. C. Miller made a Fourier analysis of the optical recordings of vowel sounds and his experimental results confirm Helmholtz's harmonic theory. He concluded that all vowels can be grouped into two classes, one having a single characteristic frequency of resonance and the other having two characteristic frequencies of resonance. The vowel sound of *a* in 'father' belongs to the former class while *e* in 'met' belongs to the latter class. I. B. Crandall with the help of a distortionless microphone and oscillographs carried out extensive study of the records of vowel sounds and could find out their characteristic frequencies and the energy distribution over the range of frequencies in the human voice. His work on the frequency characteristic of a vowel sound agrees with that of D. C. Miller.

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THE ACOUSTICS OF BUILDINGS

1. **Reverberation**: Let us consider a source continuously sounding in a closed hall. If the walls be perfectly reflecting, the energy density will soon be very large; if on the other hand the walls absorb sound perfectly, the state of affairs will be as if the walls have been removed. In that case, intensity at a point will vary inversely as the square of the distance of the point from the source. But in actual cases, there is always some reflection from the walls and the energy density in the enclosure gradually builds up to a steady state value, when the energy produced by the source per second equals the rate of loss due to dissipation. If the source of sound is suddenly cut off, still the sound persists for some time due to one or multiple reflections from the reflecting surfaces of the enclosure. This phenomenon is known as reverberation.

The reflection from the walls increases the loudness of the sound, but too much of it impairs clarity of speech in the hall. Moreover, the enclosure may resonate to particular frequencies or there may be interference causing distortion. On the other hand if reverberation is totally absent, the sonorous effect of music is affected. Thus loudness and clarity are due to opposite conditions of reflecting power.

Before 1909, designs of halls were not made from any scientific standpoint of acoustics. It was W.C. Sabine who first put the study of the subject on a scientific footing. He came to the conclusion that the requirement of a good hall or auditorium must be the following. (a) The sound heard should be loud, (b) there must be no distortion of the sound heard, or in other words, the ratio of intensities of the components of the sound made must be the same when heard, (c) successive syllables uttered must be distinctly heard.

2. Theoretical treatment of reverberation :

In an enclosure, any sound produced by the source will be reflected several hundred times before the intensity falls down to the limit of audibility. Hence, after some time we can consider the distribution of energy density to be uniform throughout the

enclosure. Let E be the energy density at any instant and δS an element of area forming the enclosure.

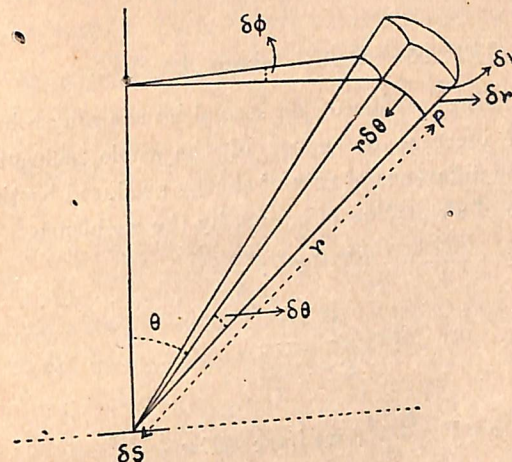


Fig. 60

Let us consider within the enclosure an element of volume $\delta v = r^2 \delta \theta \delta \phi \sin \theta$ in spherical co-ordinates at a point P distant r from δS , r making an angle θ with the normal to δS . Since the energy stream is the same in all directions, the amount of energy reaching δS from δv will be $\delta W = E \delta v \frac{\delta \omega}{4\pi}$, where $\delta \omega$ is the small solid angle subtended by δS at P .

Thus

$$\begin{aligned} \delta W &= \frac{E \delta v \delta \omega}{4\pi} \\ &= \frac{E \times r^2 \delta \theta \delta \phi \sin \theta \cdot \delta \phi \delta S \cos \theta}{4\pi r^2} \\ &= \frac{E \delta S}{4\pi} \sin \theta \cos \theta \delta \theta \delta \phi \delta r \end{aligned} \quad \dots \quad (1)$$

Now energy passing through δS from the front side will be that contained in the hemisphere of radius C , where C is the velocity of sound. Calling this E , we have

$$\begin{aligned} W &= \frac{E \delta S}{4\pi} \int_0^{\pi/2} \int_0^{2\pi} \int_0^C \sin \theta \cos \theta \, d\theta \, d\phi \, dr \quad \dots \quad (1) \\ &= \frac{1}{2} E C \delta S \end{aligned} \quad \dots \quad (2)$$

THE ACOUSTICS OF BUILDINGS

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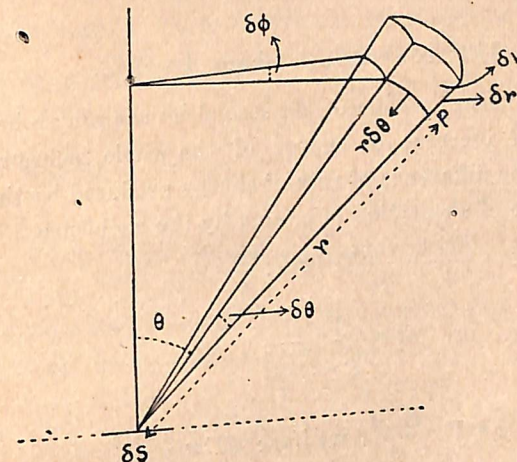


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Thus

$$\begin{aligned} \delta W &= \frac{E \delta v \cdot \delta \omega}{4\pi} \\ &= \frac{E \times r \delta \theta \cdot \delta r \cdot r \sin \theta \cdot \delta \phi \cdot \delta S \cos \theta}{4\pi r^2} \\ &= \frac{E \delta S}{4\pi} \sin \theta \cos \theta \delta \theta \delta \phi \delta r \end{aligned} \quad \dots (1)$$

Now energy passing through δS from the front side will be that contained in the hemisphere of radius C , where C is the velocity of sound. Calling this E , we have

$$\begin{aligned} W &= \frac{E \delta S}{4\pi} \int_0^{\pi/2} \int_0^{2\pi} \int_0^C \sin \theta \cos \theta d\theta d\phi dr \quad \dots (1) \\ &= \frac{1}{4} E C \delta S \end{aligned} \quad \dots (2)$$

If δS is an element of the wall of the enclosure and a the coefficient of absorption, the energy absorbed per second by the enclosure is

$$\frac{1}{4} E C \Sigma a \delta S = \frac{1}{4} E C A, \text{ where } A = \Sigma a \delta S.$$

If the energy produced per second by the source be P , then the rate of increase of energy of the whole enclosure will be equal to the difference of rate of energy produced by the source and the rate of absorption of energy by the enclosure.

$$\text{Hence } P - \frac{E C A}{4} = \frac{d}{dt}(E V) = V \cdot \frac{dE}{dt} \quad \dots \quad (3)$$

$$\text{or, } dt = \frac{V \cdot dE}{P - \frac{E C A}{4}}$$

$$\text{Writing } x = P - \frac{E C A}{4}, \text{ we have } dE = -\frac{4 dx}{C A}$$

$$\text{Hence } dt = -\frac{4 V}{C A} \frac{dx}{x}$$

Integrating within the limits $t=0$ and $t=t$, we have

$$t = -\frac{4 V}{C A} \left[\log x \right]_P^{P - \frac{E C A}{4}} \quad \dots \quad (4)$$

since at $t=0$, energy density in the enclosure is zero.

$$\text{From (4), } -\frac{C A}{4 V} \cdot t = \log \frac{P - \frac{E C A}{4}}{P}$$

$$\text{whence } \frac{1}{E} = \frac{4 P}{C A} \left(1 - e^{-\frac{C A t}{4 V}} \right) \quad \dots \quad (5)$$

The steady state energy E_0 may be obtained from (4), by putting $t = \infty$

Thus energy density in the steady state is

$$E_0 = \frac{4 P}{C A}$$

Hence we can rewrite (5) as

$$E = E_0 \left(1 - e^{-\frac{C A t}{4 V}} \right) \quad \dots \quad (6)$$

To calculate the decay of energy, let us take the help of equation (3). In this case the source has ceased to sound when the energy density in the enclosure has the maximum value E_0 .

$$\text{Hence } -\frac{E C A}{4} = V \frac{dE}{dt}$$

$$\text{whence } \frac{dE}{E} = -\frac{C A}{4 V} dt,$$

$$\therefore \left[\log E \right]_{E_0}^E = \left[-\frac{C A}{4 V} dt \right]_0^t$$

where t is the time counted from the instant the source has ceased to sound when energy density was E_0 .

$$E = E_0 e^{-\frac{C A}{4 V} t} \quad \dots \quad (7)$$

Thus

$$= \frac{4 P}{C A} e^{-\frac{C A t}{4 V}} \quad \dots \quad (8)$$

Reverberation time is defined as the time in which the energy density falls to the just audible value from an initial energy density which is 10^6 times as large. This is a range of 6 bels or 60 decibels.

$$\text{From (7) } \frac{E_0}{E} = e^{\frac{C A T}{4 V}} = 10^6$$

where T = reverberation time and E = energy density at just audible value.

$$\begin{aligned} T &= \frac{4 V}{A C} \cdot \log 10^6 \\ &= 2.3 \times \frac{4 V}{A C} \log_{10} 10^6 \\ &= \frac{2.3 \times 24 V}{A C} = \alpha \cdot \frac{V}{A} \quad \dots \quad (8) \end{aligned}$$

If the measurement is taken in feet, then by putting the velocity of sound in feet per second, we have $\alpha = .05$. In metres, the value of the constant will be equal to .16.

Optimum reverberation time : Reverberation time in a hall determines the suitability of it for speech or music. Optimum reverberation time in a hall is the most suitable reverberation time for speech or for music for which the hall is designed. The best times for particular halls, however, depend on their sizes. For a hall of about 50,000 c. ft., the optimum reverberation time lies between 1 and 1.5 seconds while for a hall of volume of 253,000 c. ft., it is about 2.3 seconds. Thus it generally has a value between 1.0 to 2.5 seconds ; smaller reverberation time is necessary for speech than for music. It must be again borne in mind that reverberation time in a hall is affected appreciably by the presence of an audience in the hall due to absorption by clothes etc.

3. Reverberation time in a dead room :

If the mean absorption coefficient a of the material of the walls of a room is less than 0.4, the room is said to be a *live* room ; it may be called a *dead* room if the mean absorption coefficient is above 0.4. For a perfectly dead room $a=1$ and the reverberation time in such a room should be according to Sabine's formula

$$T = \frac{0.05V}{S}$$

when V is in c. ft., S the area of the walls in sq. ft. and velocity is in feet per sec. Thus calculated reverberation time is not zero though actually it is so.

Taking into consideration higher order reflections in a dead room the calculated modified reverberation time T is given by

$$T = \frac{0.05V}{-S \log(1-a)}$$

when V and S are in cu. ft. and sq. ft. respectively and a the mean absorption coefficient.

Thus for a completely dead room, $a=1$ and $\log(1-a)=x=\log 0$ whence $e^x=0$ giving $x=-\infty$, and substituting in the modified expression, $T=0$

For a live room a is a small quantity ; then $\log(1-a)=-a$ and then $T = \frac{0.05V}{Sa} = \frac{0.05V}{A}$, which is Sabine's formula.

4. Measurement of absorption coefficients :

(a) Sabine measured absorption in open window units, the open window being a perfect absorber of sound. The absorption by one square foot of a material of absorption coefficient, say '6, is equivalent to that passing through '6 sq. foot of an open window. The unit of absorption coefficient is the *Sabine*.

Let us suppose, two sound sources of powers P_1 and P_2 are taken. The sources may be loudspeakers supplied by known audio frequency currents and hence of known intensities. Sabine in his experiments used organ pipes as sources.

Let us consider that the experiments are done in a reverberation chamber whose walls are covered by a material whose absorption coefficient is to be found out. Let the first source of power P_1 be sounded for a sufficient time till the energy density attains the constant maximum value $\frac{4P_1}{CA}$. Let the source be suddenly cut off and time T_1 be measured till the intensity within just reaches the limit of audibility E . Then

$$E = \frac{4P_1}{CA} e^{-\frac{CA}{4V}T_1}$$

A second similar experiment is done with the second source of power P_2 when we get, as intensity decays to same limit

$$E = \frac{4P_2}{CA} e^{-\frac{CA}{4V}T_2}$$

$$\text{Hence } \frac{P_1}{P_2} = \frac{e^{-\frac{CA}{4V}T_2}}{e^{-\frac{CA}{4V}T_1}} = e^{\frac{CA}{4V}(T_1 - T_2)}$$

$$\therefore \frac{CA}{4V}(T_1 - T_2) = \log \frac{P_1}{P_2}$$

$$\text{whence } A = aS = \frac{4V}{C} \times \frac{\log \frac{P_1}{P_2}}{T_1 - T_2} \quad \dots (9)$$

where a is the mean absorption coefficient and S the area. These experiments are to be carried out with the empty reverberation chamber in which absorption from the walls is very

small and then with the absorbent material whose coefficient of absorption is to be calculated. Sabine in his experiment expressed absorption by the absorptive material in terms of the area of an "open window." Times T_1 and T_2 in (2) can be measured by ear or by an oscillograph.

(b) **Stationary wave method :**

Absorption coefficient of a material can be measured by the use of stationary waves. The method is very convenient and gives accurate results, and a small amount of material in the form of a slab is required for the experiment. But the great defect is that the absorption coefficient can be measured only at normal incidence of the sound waves, whereas it has been experimentally verified that absorption depends to a large degree on the angle of incidence. The arrangement used by Paris consisted of an earthen ware pipe 226 cm. long and 30 cm. diameter. One end of the pipe terminated into a sound chamber in which a loudspeaker was actuated by a known current supplied by a valve oscillator at a constant frequency. The other end of the pipe was closed by the specimen of the material whose coefficient of absorption was to be found out. Stationary waves were formed inside the pipe due to superposition of the incident and partially reflected waves. A calibrated hot wire microphone was used to find out the displacements at different points along the length of the tube. If the displacements are measured with a good reflector at the end of the pipe, then by comparison of the displacements at the same points in the two cases, absorption coefficient of the slab could be calculated.

5. **Design of a hall :**

Requirements of a good auditorium are :

- (i) The reverberation time should be suitable for the purpose of music or speech for which the hall is designed.
- (ii) Resonance of section of walls, sound boards and air space in the hall should be as small as possible. The last effect is very noticeable in small halls.
- (iii) Curved walls and domed ceilings should be avoided as they tend to focus sound.
- (iv) Interference giving rise to maxima and minima of sound at different positions in the hall should be as small as possible.

The most of the defects can be eliminated by proper design of the auditorium and lagging walls with suitable sound absorbers.

6. **Study of an auditorium :**

(a) *Ripple tank method :*

A model auditorium may be tested by ripple tank method to localise the defects of a designed hall. Ripples are produced in mercury by a stylus attached to an electrically maintained tuning fork ; the stylus may be withdrawn from the mercury at any suitable instant by an electromagnet. A model auditorium is taken and the stylus is placed in the position where the source of sound should be in the actual auditorium. Waves reflected back from the walls of the model auditorium may be studied by a succession of photographs. In this experiment the wave-lengths of the ripples in the model are comparable to the wave-length of sound in the actual auditorium.

(b) *Spark pulse method :*

Spark pulse method was first used by Sabine. A model of the auditorium is placed round a spark gap which may be called "light" spark gap ; a "sound" spark is produced at the required position of the model and the progress of the pulse is photographed at varying intervals with the help of the light spark. The light spark is behind the sound spark by a fraction of a second, the exact time may be suitably adjusted. By an automatic arrangement the photographic plate is protected from the light of the sound spark. There is a great resemblance between the photography taken by the ripple tank method and the spark pulse method.

CHAPTER XVII ULTRASONICS

1. Production of ultrasonic waves : The upper limit of frequency at which sound waves cease to produce a sensation of sound in the ear depends on the individual. For convenience, frequency above 10,000 c.p.s. is called ultrasonic frequency, though waves may produce sensation of sound even when the frequency is 20,000 c.p.s. The generally used method of producing sound by supplying a loudspeaker with an audio frequency current fails in the ultrasonic range as the diaphragm or cone is incapable of vibrating at such a high frequency and the impedance of the coil is so large that no appreciable current can be passed through it. The most generally used methods to produce ultrasonic waves are (1) the magnetostriction oscillator method and (2) the piezo-electric oscillator method.

2. Magnetostriction oscillator : Whenever a rod of ferromagnetic material like iron or nickel is magnetised, it under-goes

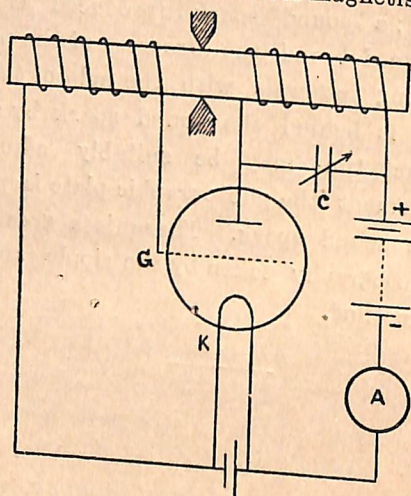


Fig. 61

a change in length. If the rod is placed in a coil through which an alternating current is passed, then in each cycle of the

magnetic field of the coil along the length of the rod, there will be two cycles of change in length, since change in length is independent of the direction of the field. But if the rod is initially magnetised by a second coil carrying direct current, the frequency of alterations in the length will be the same as that of the alternating field. The change in length of a rod due to magnetostriction is very minute ; about one part in a million ; but when the frequency of the magnetising field is the same as that of one of the natural modes of vibration, specially the fundamental one, the amplitude may be very large due to resonance, and the rod is then a powerful source of ultrasonic waves. Pure nickel and alloys of nickel like nichrome, monel metal etc., are materials for powerful magnetostriction oscillators.

A magnetostriction oscillator circuit for generation of ultrasonic waves is given in the figure. A nickel or steel rod is clamped in the middle. Two coils round the rod are in the grid and plate circuits of a triode valve are connected in such a way that by a proper feed back in the grid circuit by mutual coupling of the two coils oscillations are maintained. *C* is a variable condenser by means of which the plate circuit can be tuned to the resonant frequency of the rod. The resonance is indicated by a change of current in the milliamperemeter *A*. The component of direct current in the plate circuit through the plate coil polarises the bar magnetically, or an additional coil carrying direct current may be used.

For a rod clamped in the middle, the frequency of the fundamental vibration is given by

$$N = \frac{1}{2L} \sqrt{\frac{E}{\rho}}$$

Thus the frequency of vibration of the oscillator can be selected by a suitable choice of the length of the rod. A magnetostriction oscillator is suitable for producing ultrasonic waves up to about 60,000 c. p. s. ; for frequency above this, piezo-electric oscillators are used.

3. Piezo-electric oscillators : J. and P. Curie found out that certain asymmetric crystals like quartz, when subjected to pressure or tension on their faces, develop electric charges.

This is known as piezo-electric effect of the crystals. Rochelle salt shows the greatest piezo-electric effect, but due to its inferior mechanical properties, quartz or tourmaline is generally used. For production of ultrasonic waves quartz crystals are generally preferable. Fig. 62 represents a transverse section of a quartz crystal cut perpendicular to the optic axis of the crystal called z -axis.

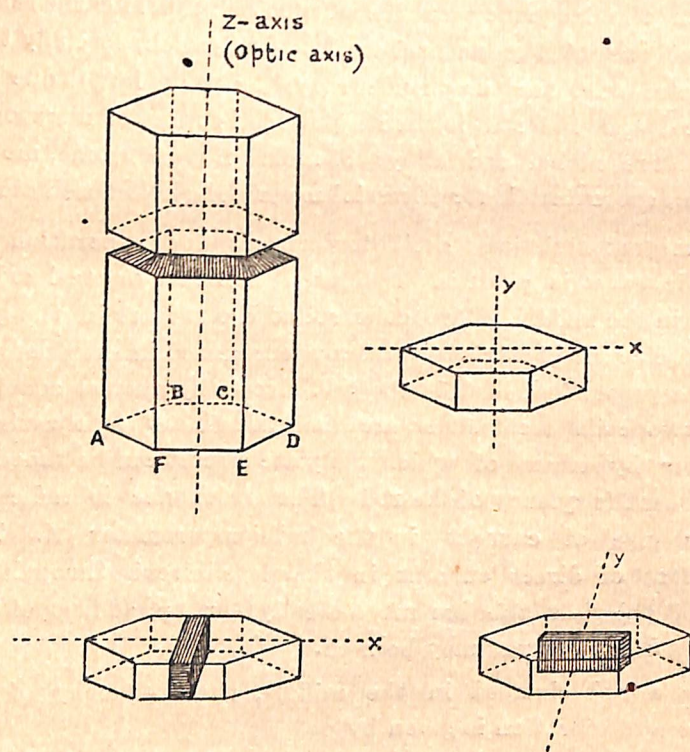


Fig. 62

Let a hexagonal slice perpendicular to the z -axis be cut. Any straight line joining two opposite angles of the hexagon may be called x -axis and a straight line perpendicular to xz plane is y -axis. The x -axes are known as electric axes of the crystal. Slices of crystals with two largest faces cut perpendicular to the x -axes or electric axes are known as x -cut crystals, while those with largest faces containing the electric axes are known as y -cut crystals.

An electric field applied in the direction of the z -axis produces no effect, while applied along x or y -axis causes the crystal to contract or expand in the direction of the two axes; expansion in one direction being associated with contraction in the perpendicular direction.

If a piezo-electric crystal is mounted within two metal plates and an alternating field is applied across them, the crystal will vibrate with a frequency equal to that of the alternating electric field. The frequencies of the possible vibrations are given by

$$N_s = \frac{s}{2l} \sqrt{\frac{E}{\rho}}$$

(See Chapter VII) where E is the Young's modulus which is different along different directions in an asymmetric crystal. The vibrations will be very vigorous if the frequency of the applied field equals one of the frequencies of the possible modes of vibration of the crystal.

A piezo-electric crystal like quartz can be maintained in vibrations with the help of a triode valve. The following circuit

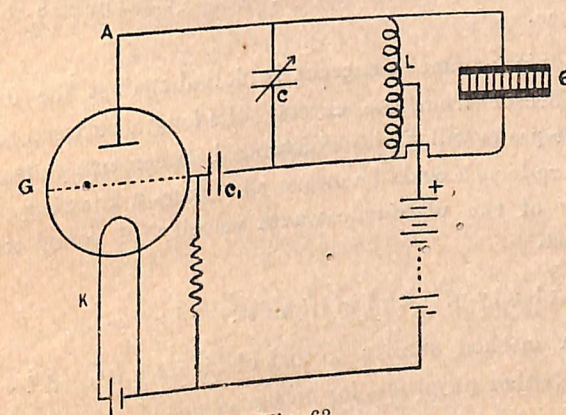


Fig. 63

known as Hartley circuit may be used with advantage. The quartz crystal is connected parallel to a tuned circuit consisting of a coil L and a variable condenser C . Proper feed back for oscillations is maintained by connecting positive terminal of the high tension battery to a tap in the coil. One end of the coil is

connected to the plate and the other to the grid through a condenser. The coil is tuned with the help of the variable condenser to the resonant frequency of the crystal. The crystal may be placed in a liquid in which strong ultrasonic waves will be produced.

4. Velocity of sound at high frequency :

(a) **Pierce's method :** The velocity of sound in air at high frequency was determined by Pierce, using a valve maintained quartz generator. The crystal surface was made accurately parallel to the surface of a reflector which could be moved parallel to itself by a micrometer screw. The waves reflected from the reflector reacted on the quartz oscillator and an ammeter placed in the plate circuit of the valve showed maxima and minima when the reflector was gradually moved towards the crystal, the change in the current depending on the phase of the reflected waves reaching the crystal. For precision measurements the ammeter could be replaced by a sensitive micro-ammeter, whose initial reading in the absence of the reflector was reduced to zero with the help of a parallel auxiliary circuit containing a coil and a resistance.

If the micro-ammeter current is plotted against the distance of the reflector from the crystal, the length between the two successive peaks will give the distance between two nodes from which wave-length could be measured. From a knowledge of the frequency of the ultrasonic waves velocity of sound could be determined.

(b) Velocity of sound in liquids :

(i) A method similar to that of Pierce was used by R. W. Wood and other physicists for measurement of velocity of sound in liquids at ultrasonic frequency. At a frequency of 2×10^5 c.p.s. or more, practically plane waves are produced by a quartz crystal of diameter about 10 cm. These waves after being reflected from a reflector reach the crystal surface and react on it as described previously. A circuit containing a neon lamp coupled with the oscillator may be extinguished when the

reflector is at a node of the stationary waves. Hence knowing the wave-length velocity in the liquid could be determined.

(ii) R. W. Boyle and Lehman measured the velocity of sound in water using two quartz transmitters. The quartz crystals in water with their faces parallel to each other are run from the same valve circuit, and hence vibrations of both are in phase. If coke dust is placed in the path of the beam, it ultimately collects at the nodes and from the nodal distance the wave-length can be determined. The velocity of sound in water as found by them was 1.52×10^5 cm./sec. approximately at 15°C .

5. Diffraction of light by ultrasonic waves :

Brilouin in 1921 predicted that light passing through a transparent liquid will be diffracted if ultrasonic waves of very short wave-length travel through the medium at the same time. Due to the effects of the ultrasonic waves, there is a sinusoidal density variation along the path of the waves and the liquid in which density variation occurs at regular distances acts as a grating, and the diffraction due to this is similar to one in X-rays by the regular spacing of atoms in a crystal. Thus if d is the grating space due to ultrasonic waves and λ the wave-length of light, then $n\lambda = 2d \sin \theta$ when θ is the angle of diffraction at the n th order. Putting $d = \lambda_0$ where λ_0 is the wave-length of the sound,

$$n\lambda = 2\lambda_0 \sin \theta = \frac{2C}{N} \sin \theta$$

where C = velocity of sound and N the frequency. According to Brilouin, when the grating constant is sinusoidal, only first order spectrum is to be expected. But in practice with a strong source of light a higher order spectrum can be obtained. A complete theory of the phenomena has been given by C. V. Raman and N. S. Nagendra Nath.

To obtain diffraction images, the following arrangement due to Debye and Sears may be made. Rays from an intense source of light rendered parallel by a lens pass through a transparent

liquid kept in a transparent vessel. By a second lens the transmitted rays are brought to a focus on a screen or a photographic plate. At the bottom of the vessel is a quartz crystal maintained in oscillation by a valve. Diffraction spectra of several orders together with the undeviated central beam may be obtained.

6. Depth sounding by ultrasonic waves : Langevin's echo depth sounder : Langevin's oscillator consists of a mosaic of small crystals of quartz 2 mm. thick cemented together and placed between two steel plates S_1 and S_2 of diameter 25 cm. The whole system vibrates as a single plate 2.2 cms. thick. The natural frequency of oscillations of the assemblage is about 40,000 c. p. s. and the corresponding wave-length in water is about 3.5 cm.

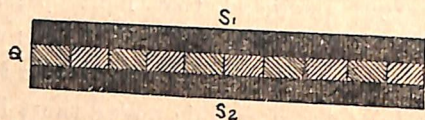


Fig. 64

One side of the arrangement is in contact with the sea, while there is a watertight box shielding the other side. High voltage oscillations are generated by a spark oscillatory circuit with which the crystal "sandwich" is coupled. A series of damped trains of waves are transmitted by the quartz crystal assemblage at controlled intervals. These ultrasonic waves are reflected from the bottom of the sea or any other obstacle whose position is to be measured and reaching the crystals generate small e.m.f. Within the time that elapses between transmission of the waves and their return to the assemblage, a switch automatically connects the oscillator to the receiving side of the circuit. The time interval between the transmission and the reception of the pulse being known from an automatic record, the depth can be found out from the knowledge of velocity of sound in sea water.

The beam sent out by oscillator forms a cone of semi-angle $\sin^{-1} \frac{1.22\lambda}{D}$ where λ is the wave-length and D the diameter of

the slab. The smaller the wave-length and greater the diameter, the less diverging will be the transmitted beam.

7. Some properties of ultrasonic waves :

(a) A rod dipped in a liquid through which ultrasonic waves are passing becomes hot very soon due to the heat generated by the vibration of the rod with the finger.

(b) Intense ultrasonic waves transform immiscible liquid like water and oil into stable emulsions. Smoke becomes co-agulated, and large particles thus formed cannot remain in suspension.

(c) Many micro-organisms are destroyed in an ultrasonic field. Small animals like fish, frogs, tadpoles etc., may be killed or maimed.

CHAPTER XVIII

TECHNICAL APPLICATIONS

1. **Microphone** : A microphone is a device which produces a periodically varying current when periodic change of pressure takes place in air due to a sound. The essential characteristics of a good microphone are the following.

- (1) The response should be uniform throughout the whole range of audio-frequencies.
- (2) The output must be free from any background noise.
- (3) The sensitivity must be high.
- (4) The instrument must be robust in construction.

(a) **Carbon microphone** : It consists of a little box containing granules of carbon between a carbon or metal diaphragm *D* and a carbon plate *C*. Pressure change in air due to sound waves in front of the microphone causes the diaphragm to vibrate exerting a varying pressure on the granules. The electrical resistance of the contacts varies according to the periodic pressure change ; and a fluctuating current is obtained when terminals of a battery are connected to the diaphragm and the carbon plate. The electrical circuit may contain the primary coil of a transformer and output may be obtained from the secondary.

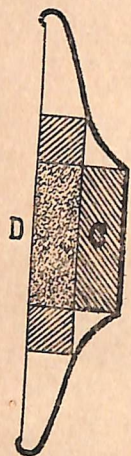


Fig. 65

The diaphragm is usually damped by flannel or cotton wool washers to avoid resonance at particular frequencies. The microphone is very sensitive but has a background hiss, specially when increased output is obtained employing batteries with larger voltages. The hiss is due to local heating of the carbon granules.

The quality of output of a carbon microphone is rather poor due to unavoidable response to certain frequencies and an

unequal change of resistance in equal displacements of the diaphragm in forward and backward directions.

Current *i* through a carbon microphone can be expressed as

$$i = \frac{e}{R + r \sin pt} = \frac{e}{R} \left(1 - \frac{r}{R} \sin pt + \frac{r^2}{R^2} \sin^2 pt \dots \right)$$

where *e* = e.m.f of the battery, *R* the steady component and *r sin pt* the fluctuating component of the resistance of the circuit. The third term within the bracket shows a second harmonic distortion. Neglecting it

$$i = \frac{e}{R} \left(1 - \frac{r}{R} \sin pt \right)$$

If change in resistance is proportional to displacement *x* of the diaphragm, then the fluctuating part *i*₁ of the current is

$$i_1 = -\frac{er}{R^2} \sin pt$$

$$= -\frac{ek}{R^2} x, \text{ where } k \text{ is a constant.}$$

Thus if the displacement is proportional to sound pressure, the output will be distortionless.

(b) **Electro-dynamic microphones** : In this type current is generated due to movement of a coil or a conductor perpendicular to a magnetic field. The induced e.m.f. being equal to rate of change of flux is proportional to the velocity of the conductor and if the fluctuating sound pressure is proportional to the velocity, the output will be distortionless. The microphones belonging to this class may be called "velocity" microphones. If *F sin pt* be the fluctuating sound pressure, the motion of the moving part of the microphone can be written as

$$F \sin pt = m \cdot \frac{d^2 x}{dt^2} + k \cdot \frac{dx}{dt} + sx,$$

(i) If *k* and *s* are negligible, then

$$\frac{dx}{dt} = -\frac{F}{mp} \cos pt;$$

thus output is inversely proportional to frequency.

(ii) If m and k are negligible, then

$$\frac{dx}{dt} = \frac{pF}{s} \cos pt$$

Hence output increases with frequency.

(iii) If damping (whether electromagnetic or mechanical) is very great,

$$\frac{dx}{dt} = \frac{F}{k} \sin pt.$$

Thus in the 3rd case we get uniform frequency response. The sensitiveness is, however, impaired by excessive damping.

2. (a) Moving coil microphone :

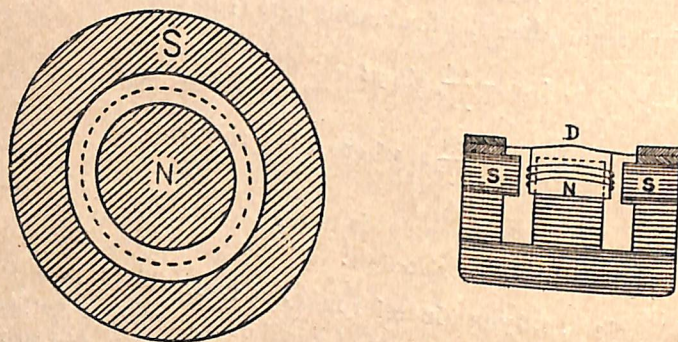


Fig. 66

D is a rigid thin diaphragm with which is attached a coil which can move freely in the annular clearance between the north pole and south pole of a pot magnet whose section is shown in the figure. The field is radial and the movement of the coil causes generation of an induced e.m.f. which can be amplified by a valve amplifier. The response of the above type of microphone is almost linear.

2. (b) Ribbon microphone :

This consists of a very thin ribbon R of corrugated aluminium or aluminium alloy suspended in a magnetic field parallel to its plane. The ribbon is exposed to sound waves both

from front and back and hence pressure on it is the difference of pressures acting from both sides. The induced e.m.f. generated by

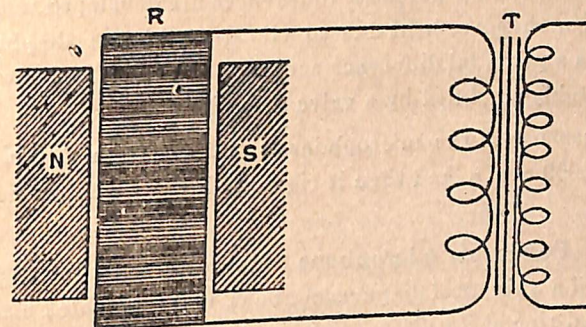


Fig. 67

the vibration of the ribbon may be stepped up by a matched transformer T . The electromagnetic damping is very great.

The sensitivity of the above type of microphones is less than that of the moving coil microphones, but the frequency response is flat. The response of a ribbon microphone is directional.

2. (c) Crystal microphone : If alternating pressures are applied to the faces of a piezo-electric crystal, alternating potential difference is developed between the faces. This is most marked in Rochelle salt and best effects are produced when alternating sound pressure twists the crystal about its principal axis.

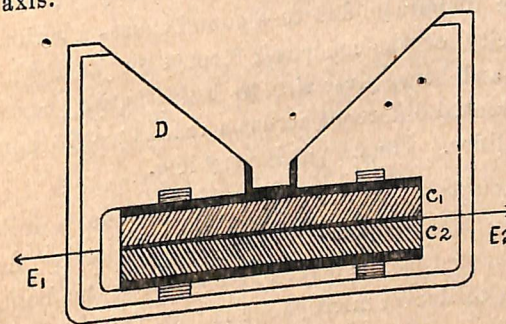


Fig. 68

Two thin slices are cut from the same crystal and cemented together. Between the two slices and on the surfaces of the

two crystals, there are thin metal foils which form the electrodes as shown in the figure. The slices are cut in such a manner that equal and similar potential differences are developed in each. Hence when they are bent due to vibration of the diaphragm D , there is a potential difference across the two electrodes which can be suitably amplified by a valve amplifier.

The response of the microphone is uniform from 100 c. p. s. to about 5000 c. p. s. and then it rises gradually. The sensitivity is poor.

2. (d) **Condenser microphone**: A condenser microphone consists of a thin steel diaphragm about .002 inch thick clamped by a metal ring at its edges. P is another metal plate facing D , and the air space between the two is only about .001 inch. The two are insulated from each other and output is taken from the leads L_1 and L_2 .

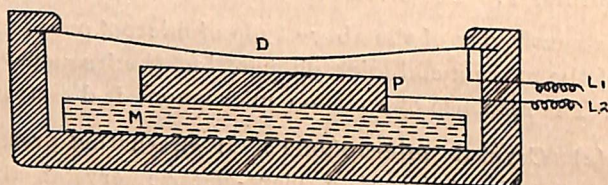


Fig. 69

A steady potential difference of about 300 volts is applied across D and P through a high resistance. Any periodic displacement of the diaphragm due to a sound causes a periodic change in the capacity of the condenser formed by the diaphragm and the plate, and hence gives rise to a fluctuating current. The varying potential difference across a resistance is amplified by a valve amplifier. The output is very low.

The response of a condenser microphone is uniform from 50 c. p. s. to 8000 c. p. s. The linear response due to the damping is caused by the layer of air between the diaphragm and the plate. A condenser microphone is very suitable for measuring intensity of a complex sound.

Action of a condenser microphone can be understood from the following.

Let C_0 be the capacity of a condenser microphone when undisturbed by sounds, E the e.m.f. of cells, r a very high resistance, all in series.

Suppose a sound of frequency $p/2\pi$ is made in front of the microphone and let us suppose that the simple harmonic pressure changes in air cause corresponding changes in the capacity of the condenser microphone which may, at any instant, be written as

$$C = C_0 + \delta C = C_0 + C_1 \sin pt$$

The small increase of charge due to the change in the capacity is $E\delta C$. Hence the simple harmonic potential difference developed is

$$\frac{E\delta C}{C}$$

The instantaneous current may be written as

$$i = \frac{E\delta C}{C\sqrt{r^2 + \frac{1}{C_0^2 p^2}}} = \frac{EC_1 \sin(pt + \alpha)}{C_0\sqrt{r^2 + \frac{1}{C_0^2 p^2}}}$$

[where $\tan \alpha = \frac{1}{Cpr}$ and $C - C_0$ is a small quantity]

Hence p. d. across r is

$$\frac{rEC_1 \sin(pt + \alpha)}{C_0\sqrt{r^2 + \frac{1}{C_0^2 p^2}}}$$

If $r \gg \frac{1}{C_0 p}$ all components of a complex vibration will be equally amplified.

2. Loudspeaker : moving coil type :

The principles of a loudspeaker can be understood from a study of the figure. To the apex of the cone of a stiff piece of paper, a cylindrical paper former carrying a coil of several turns of thin copper wire is attached. This former can move freely within the annular space between the poles of a pot magnet when the cone vibrates. The cone is generally supported at its periphery by a cardboard ring.

When amplified fluctuating current is passed into the coil, there is a force on the coil due to the radial magnetic field.

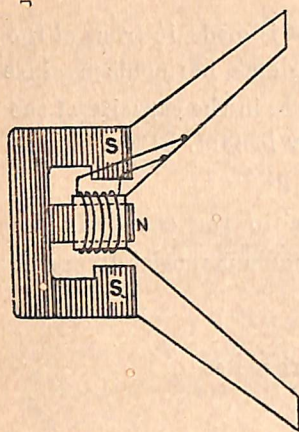


Fig. 70

This force is liH where l is the total length of the wire constituting the coil, i the current through it and H the magnetic field. Hence the coil will vibrate along its axis due to the fluctuating force. The paper cone, being attached to it, will also vibrate in a similar manner and generate sound waves.

The construction of a moving coil loudspeaker is similar to that of a moving coil microphone. The former transforms fluctuating current into mechanical vibrations, while the latter generates current when mechanical vibrations are produced in air. Thus a loudspeaker is a microphone working backwards.

The magnet used may be either an electromagnet or a permanent magnet. When the cone vibrates, if compressions are produced in the front, there will be rarefactions behind; thus there is a difference of phase between sounds coming simultaneously from the two sides. Hence a large baffle is used to screen vibrations coming from behind.

3. Recording of film: A standard size film is 35 mm. wide and the width of the space for recording sound at the edge is nearly 2.5 mm. Generally two methods of recording sound are in use: (a) variable density method, (b) variable area method.

(a) Variable density method: One method consists in modulating the brightness of a gas discharge lamp by amplified microphone currents. A slit illuminated by this light will have an image of varying brightness. Thus the sound track in this method will consist of parallel transverse lines of varying opacity according to the brightness of the image of the slit on the film.

In the modern variable density method a light "valve" is used.

The light valve consists of a shutter of a duralumin loop in front of a slit. Light from a bright source is allowed to fall on a slit whose image is formed on the film. A strong magnetic field

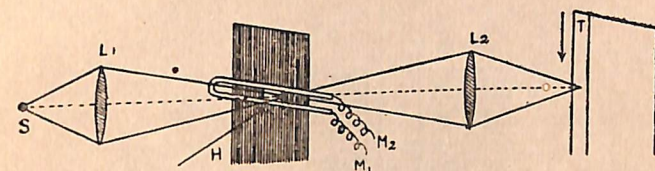


Fig. 71

by an electromagnet is applied transverse to the loop which vibrates when amplified speech current flows through the loop. Since current flows in opposite directions through the upper and lower portions of the loop, they will either approach each other or recede from each other, thus permitting less or more light to pass through the slit. The negative of the sound track on the film, therefore, will consist of lines of different opacities.

(b) Variable width method:

Rays of light from a source after passing through a lens are reflected from the surface of an oscillograph mirror to a slit behind which the film is moved in the direction of the arrow. Amplified microphone currents pass through the leads M_1 and M_2 . Due to the action of the magnetic field on the loop carrying the fluctuating current, the mirror will vibrate about a vertical axis passing through its middle point. The extent to which the length of the slit will be illuminated will depend on the orientation of the mirror which is again determined by the strength of the current flowing through the loop. Thus on the film the image of only a part of the length of the slit will be formed. The negative of the sound track will, therefore, consist

of a dark portion which is due to the image of the slit and a

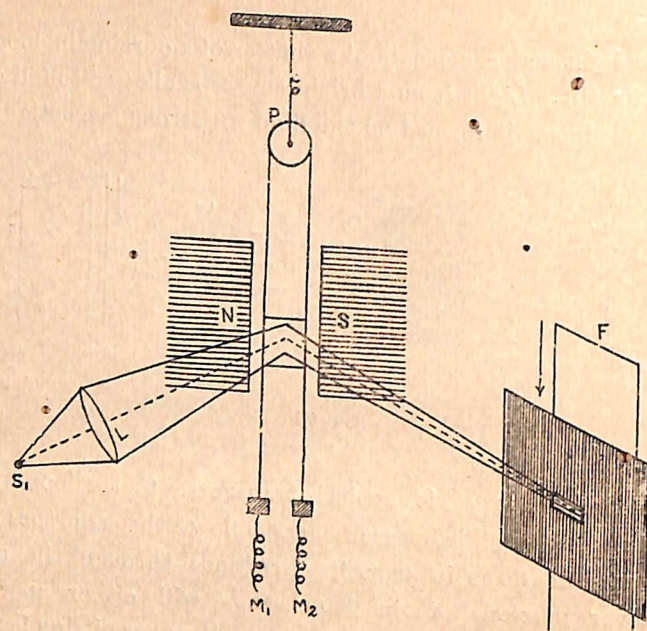


Fig. 72

transparent unaffected portion. The appearances of the sound track in the two methods are given in the figure.

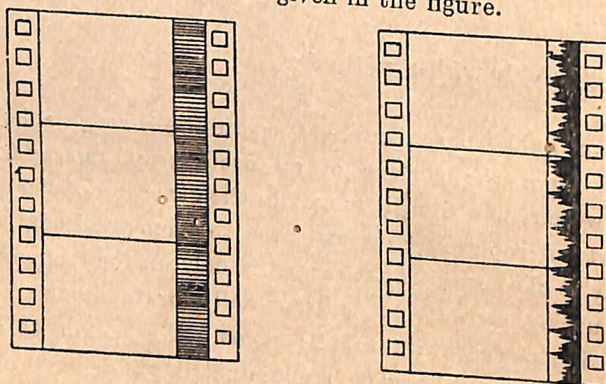


Fig. 73

4. Reproduction of sound :

For both types of records, the method applied for reproducing sound is the same. Light from an exciter lamp passes through

the sound track and is focussed on the cathode of a photo-electric cell. The photo-electric current generated is proportional to the incident intensity. Thus as the film moves across the beam at the rate at which recording was made in the circuit, we shall

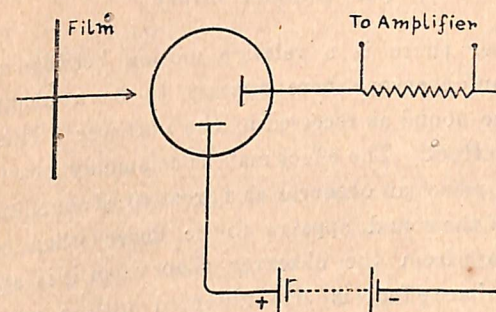


Fig. 74

get a fluctuating current due to fluctuating brightness of the beam on the cathode. The varying potential difference across a high resistance in the circuit of the photo-electric cell is amplified by a valve amplifier, and then the amplified current is passed into a loudspeaker.

APPENDIX

(A) Doppler Effect

Whenever there is a relative motion between a source of sound and an observer, there appears to be a change in the pitch of the sound as received by the observer. This is known as **Doppler effect**. The effect may be commonly observed when a motor car passes an observer at a great speed sounding its horn. The pitch of the sound appears to be lower when the car is moving away from the observer than when it is approaching him. A similar effect is also observed with a passing train blowing its whistle.

In the following detailed discussion on Doppler effect, we assume the velocity of the source or the observer to be smaller than that of sound in the medium.

Case (i). Source in motion towards a stationary observer :

Let the source be moving with a velocity u towards a stationary observer at O . In the time the source has moved from S_1 to S_2 , let us suppose it has executed one complete vibration and

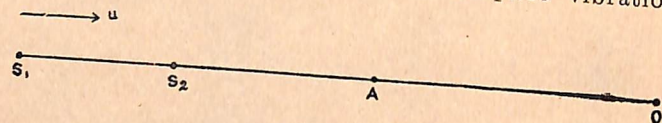


Fig. 75

the disturbance in the medium (air) created at S_1 has moved to A . If T be the periodic time of vibration of the source and c the velocity of sound in air, $S_1 A$ is equal to cT . Since S_2 and A are the successive positions where the displacements of the air particles are the same when source is in motion, the changed wave-length $\lambda' = S_2 A$. Since the velocity of sound is independent of the motion of the source, the number of complete waves received by the observer per second will be

$$N' = \frac{c}{\lambda'}$$

$$\text{But } \lambda' = S_2 A = S_1 A - S_1 S_2 = cT - uT$$

$$= \frac{c-u}{N} \quad \text{where } N = \text{frequency of the source}$$

$$\therefore N' = \frac{c}{\lambda'} = \frac{c}{c-u} N$$

$$= N \cdot \frac{c}{c-u} \quad \dots \quad (1)$$

Thus the pitch of the sound to the observer will appear to be higher than the actual pitch of the source when it is moving towards the observer. If the source is moving away from the observer, substituting $-u$ for u in (1), we shall get

$$N' = N \cdot \frac{c}{c+u} \quad \dots \quad (2)$$

Thus the apparent pitch will be lower in this case.

Case (ii). Source stationary, observer in motion away from the source :

Let S be the position of source, O_1 the initial position of the observer and O_2 that after one second. In this time the observer

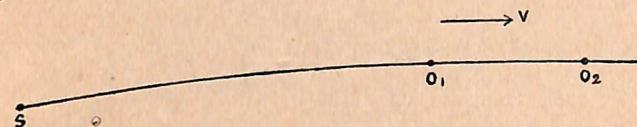


Fig. 76

will receive all the waves contained in a length c except those contained in $O_1 O_2$. Thus the number of waves received per second by the observer will be

$$N' = \frac{c-v}{\lambda}$$

where v is the velocity of the observer, we can write

$$N' = \frac{N(c-v)}{N\lambda} = \frac{N(c-v)}{c} \quad \dots \quad (3)$$

Thus the apparent pitch is lower than the actual pitch in this

case. If the observer were in motion towards the source, substituting $-v$ for v , the apparent pitch would be

$$N' = \frac{N(c+v)}{c} \quad (4)$$

and the observed pitch would be higher.

In this case the wave-length of the disturbance has not changed at all, whereas in case (i) there has been an actual change in the wave-length; the successive positions of identical displacements at any instant in the medium are different from what they are when the source is stationary.

Case (iii). Source and observer in motion along the same straight line :

Let u and v be the velocities of the source and the observer respectively in the same direction along the same straight line.

Let λ' = changed wave-length in air due to the motion of the source. Then $\lambda' = \lambda - \frac{u}{N}$ where λ is the wave-length when source is stationary. If N_1 is the apparent frequency due to the velocity of the receiver, then from the previous discussions,

$$\begin{aligned} N_1 &= \frac{c-v}{\lambda'} \\ &= (c-v) / \left(\lambda - \frac{u}{N} \right) \\ &= \frac{N(c-v)}{N\lambda - u} \end{aligned} \quad (5)$$

If the wind blows with a velocity w in the same direction, we must put $c+w$ for c and then get for the apparent pitch

$$N' = \frac{N(c+w-v)}{c+w-u} \quad (6)$$

Case (IV). Source moving in any direction, observer stationary :

At any instant t , let the source be at A moving along AB with a velocity u in the direction of the arrow, while the observer

is stationary at O . Let the source be at B after time δt . A disturbance created by the source at A will reach the observer

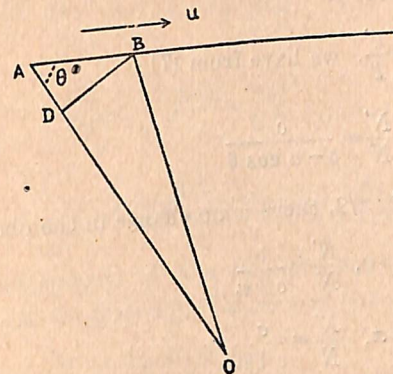


Fig. 77

at O at time $t + \frac{OA}{c}$, where c = velocity of sound. The disturbance created at B at time $t + \delta t$ will reach O at time $t + \delta t + \frac{OB}{c}$.

Hence the time that elapses between receptions at O of disturbances created by the source at A and B at times t and $t + \delta t$ is

$$\begin{aligned} &t + \delta t + \frac{OB}{c} - \left(t + \frac{OA}{c} \right) \\ &= \delta t - (OA - OB)/c \\ &= \delta t - \frac{AD}{c}, \text{ } B'D \text{ being a perpendicular on } OA. \end{aligned}$$

Hence if T be the time period of the source and T' the time for receiving a complete vibration by the observer

$$\begin{aligned} \frac{T}{T'} &= \frac{\delta t}{\delta t - \frac{AD}{c}} \\ &= \frac{\delta t}{\delta t - \frac{u \cos \theta \delta t}{c}} \\ &= \frac{c}{c - u \cos \theta} \end{aligned} \quad (7)$$

θ being the angle made by the straight line joining the positions of the source and the observer with the direction of motion of the source.

Since $\frac{N'}{N} = \frac{T'}{T}$, we have from (7)

$$\frac{N'}{N} = \frac{c}{c - u \cos \theta} \quad \dots (8)$$

Thus when $\theta = \pi/2$, there is no change in the observed pitch.

If $\theta = 0$, $\frac{N'}{N} = \frac{c}{c - u}$

and if $\theta = \pi$, $\frac{N'}{N} = \frac{c}{c + u}$

as found earlier.

Case (V). Source stationary, observer moving in any direction :

Let the source be stationary at S and observer be moving along AB with a velocity v . Let A be the position of the observer at any instant and B that after a very short interval of time δt .

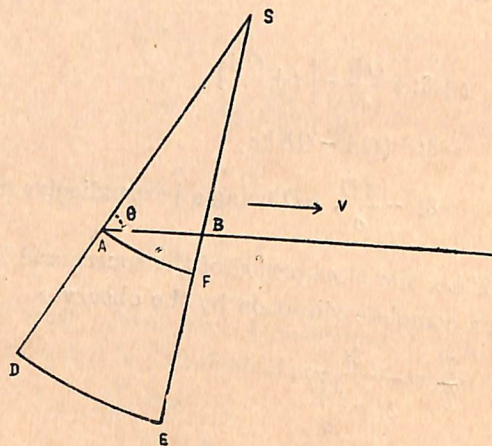


Fig. 78

In this time the wave that was at A and F initially has moved to positions D and E , so that $SD = SE$. If the observer were

stationary, he would have received the waves contained in AD , but as he has moved in the mean time to B , the waves contained in BF will also pass across him. Thus if N' is the apparent frequency

$$\begin{aligned} \frac{N'}{N} &= \frac{AD}{\lambda} \bigg/ \frac{BE}{\lambda} = \frac{AD}{BE} \\ &= \frac{AD}{AD + BF} = \frac{AD}{AD + AB \cos \theta} \quad \dots (9) \end{aligned}$$

where θ = angle made by the straight line joining source and observer with the direction of motion of the observer.

$$\begin{aligned} \text{From (9)} \quad \frac{N'}{N} &= \frac{c \delta t + \delta t \cdot v \cos \theta}{c \delta t} \\ &= \frac{c + v \cos \theta}{c} \quad \dots (10) \end{aligned}$$

If $\theta = \pi/2$, $N' = N$ and there is no change in the apparent pitch.

If $\theta = 0$, $\frac{N'}{N} = \frac{c + v}{c}$

and if $\theta = \pi$ $\frac{N'}{N} = \frac{c - v}{c}$

and we get results obtained earlier in case (ii).

Case (V). Source and observer moving in any direction :

If the source and the observer are moving in any direction, so that the direction of motion of the source makes an angle θ_1 with the straight line joining the source and the observer at any instant and the direction of motion of the observer makes an angle θ_2 with the same line at the same instant, then the apparent pitch is given by

$$N' = N \cdot \frac{(c - v \cos \theta_2)}{c - u \cos \theta_1}$$

by substituting $v \cos \theta_2$ for v and $u \cos \theta_1$ for u in (5).

(B) Melde's Experiments

F. Melde in 1859 showed experimentally that the transverse vibrations of a string under tension can be maintained by a vibrating tuning fork. The string, one end of which is attached to the tip of a tuning fork, passes over a pulley and carries a weight W at the other end.

1. Transverse arrangement of the fork :

In this arrangement as shown in the figure, the prongs vibrate transverse to the direction of the string. By adjusting the length of the string or by changing the weight W ,

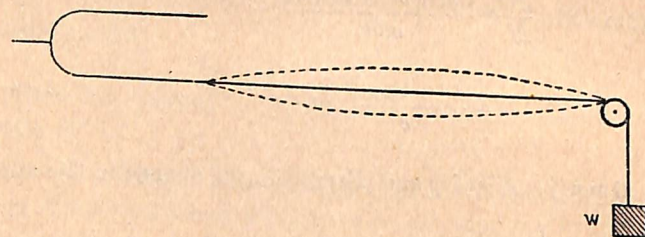


Fig. 79

strong resonant stationary vibrations may be generated in the string which may vibrate in one or several loops. If x = length of a loop between two nodes, then $v = \lambda/2$ or $\lambda = 2x$. If c is the velocity and N the frequency of transverse vibrations of the string which is also the frequency of vibrations of the fork, then

$$c = N\lambda = N \cdot 2x = \sqrt{\frac{T}{m}},$$

where T is the tension and m mass per unit length of the string.

If l = length of the string, then $l = sx$ where s is an integer.

$$\therefore N = \frac{1}{2x} \sqrt{\frac{T}{m}} = \frac{s}{2l} \sqrt{\frac{T}{m}}$$

The laws of length, tension and mass can be verified with the help of the above relation.

2. Longitudinal arrangement of the fork :

In this arrangement, the tip of the fork vibrates along the string. By adjusting length and tension, the string may be

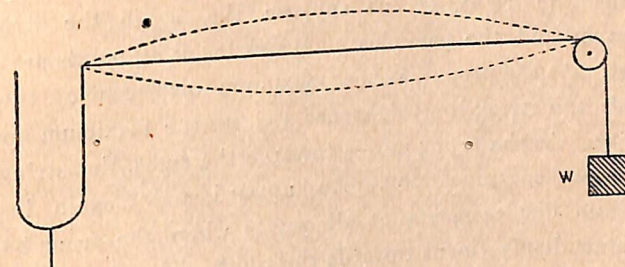


Fig. 80

made to vibrate with a large amplitude. It is then seen that the frequency of the string is half that of the fork. Thus it is an example of subharmonic resonance.

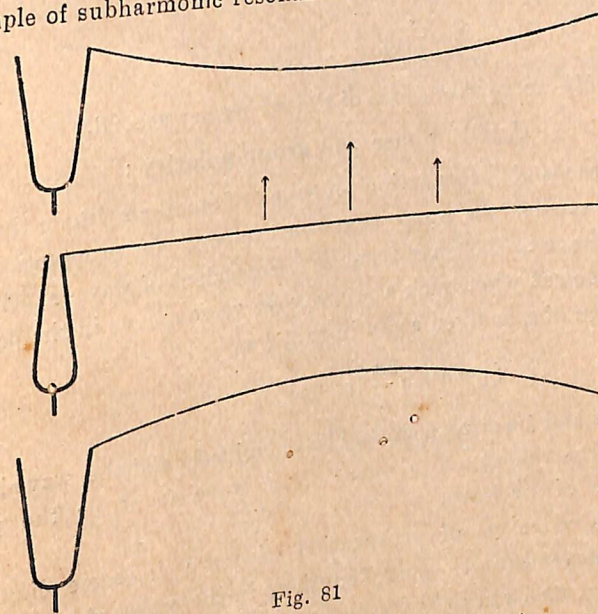


Fig. 81

The string can, however, be made to vibrate in more than one loop and if tension is kept constant, the distance between two consecutive nodes are always the same.

Lord Rayleigh worked out the theory of this class of resonance. He considered that the vibration of the fork created

periodic change of tension in the string. A simple explanation of the vibration can, however, be given in the following way.

When the prong of the fork has the maximum displacement towards the right, let us suppose the sag in the string is maximum and the particles of the string at all points are then at rest. As the prong has the maximum displacement towards the left, the string is straightened and has the maximum upward velocity. As the prong moves towards the right, the string due to inertia continues to move upward and again has the maximum displacements at all points when the prong has the maximum displacement towards the right. Thus in one complete vibration of the fork the string has undergone only half a vibration.

The laws of transverse vibrations of the string can also be verified in this case, since

$$N = \frac{3}{2l} \sqrt{\frac{T}{m}}$$

where N = no. of vibrations of the string per second.

✓(C) Phase and group velocity :

In cases of propagation of sound of small amplitude through elastic media, we know that the velocity of propagation of a disturbance in the wave-front is independent of wave-length. The displacement when such a progressive wave moves in the positive direction of x is given at an instant t as

$$\xi_1 = a \sin \frac{2\pi}{\lambda} (ct - x)$$

when a and λ are respectively the amplitude and the wave-length and c the velocity of a phase in a wave front or the phase velocity of the wave.

The velocity of propagation of wave in some dispersing media depends on the wave-length as in the cases of ripples on a liquid surface and light waves (electro-magnetic waves) through a transparent medium. In such a case some of the simple harmonic components which the wave system is built of will gain on the others and the resultant wave form will change so that the resultant amplitude moves with a velocity called group velocity which is different from those of the phase waves.

Let us have two progressive waves of same amplitude proceeding along x given by

$$\xi_1 = a \sin \frac{2\pi}{\lambda} (ct - x) \quad \text{and}$$

$$\xi_2 = a \sin \frac{2\pi}{\lambda + d\lambda} \{(c + dc)t - x\}$$

one train of waves having slightly larger phase velocity and larger wave-length than the other.

The resultant displacement is then $\xi = \xi_1 + \xi_2$

$$\begin{aligned} \text{Now } \xi_2 &= a \sin \frac{2\pi}{\lambda} \left[\left(1 - \frac{d\lambda}{\lambda}\right) \{(c + dc)t - x\} \right] \\ &= a \sin \frac{2\pi}{\lambda} \left[(ct - x) - \left\{ \frac{d\lambda}{\lambda} (ct - x) - tdc \right\} \right] \end{aligned}$$

Hence $\xi = \xi_1 + \xi_2$

$$\begin{aligned} &= a \sin \frac{2\pi}{\lambda} (ct - x) \left[1 + \cos \frac{2\pi}{\lambda} \left\{ \frac{d\lambda}{\lambda} (ct - x) - tdc \right\} \right] \\ &\quad - a \cos \frac{2\pi}{\lambda} (ct - x) \sin \frac{2\pi}{\lambda} \left\{ \frac{d\lambda}{\lambda} (ct - x) - tdc \right\} \end{aligned}$$

$$= A \sin \frac{2\pi}{\lambda} (ct - x - \theta)$$

$$\text{where } \tan \frac{2\pi}{\lambda} \theta = \frac{\sin \frac{2\pi}{\lambda} \left\{ \frac{d\lambda}{\lambda} (ct - x) - tdc \right\}}{1 + \cos \frac{2\pi}{\lambda} \left\{ \frac{d\lambda}{\lambda} (ct - x) - tdc \right\}}$$

$$\begin{aligned} \text{and } A^2 &= 2a^2 \left[1 + \cos \frac{2\pi}{\lambda} \left\{ \frac{d\lambda}{\lambda} (ct - x) - tdc \right\} \right] \\ &= 2a^2 \left[1 + \cos \frac{2\pi d\lambda}{\lambda^2} \left\{ \left(c - \lambda \frac{dc}{d\lambda} \right) t - x \right\} \right] \\ &= 4a^2 \cos^2 \frac{\pi d\lambda}{\lambda^2} \left\{ \left(c - \lambda \frac{dc}{d\lambda} \right) t - x \right\} \end{aligned}$$

For small values of x since dc and $d\lambda$ are small, we see $\tan \theta \rightarrow 0$ and the resultant disturbance is given by

$$\xi = A \sin \frac{2\pi}{\lambda} (ct - x)$$

$$\text{where } A = 2a \cos \frac{\pi d\lambda}{\lambda^2} \left\{ \left(c - \lambda \frac{dc}{d\lambda} \right) t - x \right\}$$

It is clear from the expression that the resultant amplitude has a velocity v given by $v = c - \lambda \frac{dc}{d\lambda}$; also the amplitude changes simple harmonically with x and t .

Thus the group velocity v is obtained in terms of the phase velocity and the wave-length of the phase waves.

As regards energy, it will belong to the group rather than the individual waves constituting the group.

(D) Sharpness of resonance

In page 29, we have

$$\frac{E}{E_m} = \frac{4b^2}{n^2 \Delta^2 + 4b^2}$$

The slope of a curve in fig. 10 is

$$\frac{d\left(\frac{E}{E_m}\right)}{d\Delta} = -\frac{8b^2 n^2 \Delta}{(n^2 \Delta^2 + 4b^2)^2} \text{ when } n \text{ is constant.}$$

$$\begin{aligned} \text{Hence } \frac{d^2\left(\frac{E}{E_m}\right)}{d\Delta^2} &= -\frac{8b^2 n^2}{(n^2 \Delta^2 + 4b^2)^2} + \frac{32b^2 n^4 \Delta^2}{(n^2 \Delta^2 + 4b^2)^3} \\ &= -\frac{n^2}{2b^2} \text{ when } \Delta = 0 \text{ at resonance.} \end{aligned}$$

Now radius of curvature R is given by

$$\frac{1}{R} = \frac{\frac{d^2 y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

$$\text{When } \Delta = 0, \text{ we have } \frac{dy}{dx} = \frac{d\left(\frac{E}{E_m}\right)}{d\Delta} = 0$$

Hence curvature $\frac{1}{R}$ of a curve in fig. 10 will be, at resonance i. e., is at $\Delta = 0$, given by

$$\frac{1}{R} = -\frac{n^2}{2b^2}$$

Smaller the value of R , sharper is the resonance. Thus sharpness will be very great if n is very large and b very small.

(E) Acoustic Impedance

Suppose we have a vibrating column of air in a conduit; let us consider instantaneous displacement over a section of area A to be uniform and also the length of the conduit small in comparison with the wave-length of sound, so that there is no appreciable difference of phase over the length. If m be the mass of air and ξ displacement of a particle at any instant t , we can write the equation of motion of the air under a periodic force as

$$m \frac{d^2 \xi}{dt^2} + s\xi = F e^{i\nu t} \quad \dots (1)$$

Writing volume displacement $X = A\xi$, we have

$$\frac{m}{A^2} \frac{d^2 X}{dt^2} + \frac{s}{A^2} X = \frac{F}{A} e^{i\nu t} = P e^{i\nu t} \quad \dots (2)$$

where P is the amplitude of the applied periodic pressure over the section.

Writing $\frac{m}{A^2} = l$ and $\frac{s}{A^2} = \frac{1}{c}$, equation (2) can be rewritten as

$$l \frac{d^2 X}{dt^2} + \frac{X}{c} = P e^{i\nu t} \quad \dots (3)$$

Putting $\frac{dX}{dt} = \dot{X} e^{i\nu t}$ and simplifying

we have

$$\frac{dX}{dt} = \frac{P e^{i\nu t}}{i\left(lp - \frac{1}{c}\right)} = \frac{P e^{i(pt - \pi/2)}}{lp - \frac{1}{cp}} \quad \dots (4)$$

The above equation is similar to that for an instantaneous electrical current j through an inductance l and capacitance c under an applied periodic electromotive force in the circuit given by

$$j = \frac{E_0 e^{i\nu t}}{i\left(lp - \frac{1}{cp}\right)}$$

From (4), we get acoustic impedance as $z = lp - \frac{1}{cp}$; $l = \frac{m}{A^2}$ is known as *inertance* and $c = \frac{A^2}{s}$ as *compliance*, these being equivalent to inductance and capacitance in electrical circuits. The mass reactance $\frac{m}{A^2}p$ due to inertance and reactance $\frac{s}{A^2 p}$ due to compliance act in opposition to each other. We can write the combined impedance $z = z_1 + z_2$ where $z_1 = \frac{mp}{A^2}$ and $z_2 = -\frac{s}{A^2 p}$.

When $z = 0$, i.e., $z_1 = z_2$ or $lp = \frac{1}{cp}$, the rate of volume displacement is maximum. The frequency in that resonance condition is $N = \frac{p}{2\pi} = \frac{1}{2n\sqrt{lc}}$.

(E) Acoustic Filters

By proper adjustment of acoustic circuits, an acoustic filter can be made to transmit desired ranges of frequencies.

Let us consider several conduits in series each of impedance z_1 separated by branches in parallel each of impedance z_2 . Let each conduit be so small in dimension that no appreciable phase difference occurs in it.

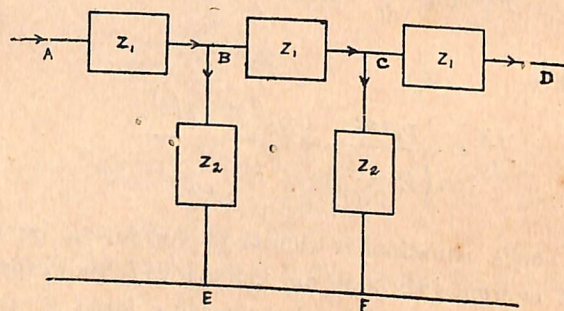


Fig. 82

Let X_1, X_2, X_3 be rates of volume displacement in AB, BC, CD respectively.

Then applying Kirchhoff's Law in the closed-acoustic circuit BEFCB

$$(\dot{X}_1 - \dot{X}_2)z_2 - (\dot{X}_2 - \dot{X}_3)z_2 - \dot{X}_3 z_1 = 0$$

$$\text{whence } \frac{\dot{X}_1}{\dot{X}_2} + \frac{\dot{X}_3}{\dot{X}_2} = 2 + \frac{z_1}{z_2}$$

We may suppose that rate of volume displacement decreases uniformly over each series circuit due to similar impedance in shunt after each conduit.

$$\text{Writing } \frac{\dot{X}_1}{\dot{X}_2} = \frac{\dot{X}_3}{\dot{X}_2} = e^\alpha$$

$$\text{We have } e^\alpha + e^{-\alpha} = 2 + \frac{z_1}{z_2}$$

$$\text{or } 1 + \frac{z_1}{2z_2} = \frac{e^\alpha + e^{-\alpha}}{2} = \cos h\alpha$$

If α is imaginary, there will be no attenuation during transmission through the line. Let $\alpha = i\beta$, where β is real. Then $\cos h\alpha = \cos i\beta = \cos \beta$. Since $\cos \beta$ can have a value between +1, and -1, α will be imaginary between those limits. Thus there will be no attenuation if $1 + \frac{z_1}{2z_2}$ lies between +1 and -1

i.e., when $\frac{z_1}{z_2}$ lies between 0 and -4.

High Pass Filter

$$\text{Let } z_1 = -\frac{1}{pc_1} \text{ and } z_2 = pl_2$$

$$\text{When } \frac{z_1}{z_2} = 0, \frac{1}{p^2 c_1 l_2} = 0 \text{ or } p = \infty.$$

$$\text{When } \frac{z_1}{z_2} = -4, \frac{1}{p^2 c_1 l_2} = 4 \text{ or } p = \frac{1}{2\sqrt{c_1 l_2}}.$$

Hence this filter will pass frequencies from $\frac{1}{4\pi\sqrt{c_1 l_2}}$ to infinity.

Low Pass Filter

$$\text{Let } z_1 = l_1 p, \quad z_2 = -\frac{1}{pc_2}$$

$$\text{For } \frac{z_1}{z_2} = 0, \quad l_1 c_2 p^2 = 0 \text{ or } p = 0$$

$$\text{For } \frac{z_1}{z_2} = -4, \quad -l_1 c_2 p^2 = -4$$

$$\text{or } p = \frac{2}{\sqrt{l_1 c_2}}$$

Thus the above filter will transmit unattenuated frequencies between 0 and $\frac{1}{\pi \sqrt{l_1 c_2}}$

Band Pass Filter

$$\text{Let } z_1 = l_1 p - \frac{1}{pc_1} \text{ and } z_2 = l_2 p - \frac{1}{pc_2}$$

$$\text{For } \frac{z_1}{z_2} = 0, \quad \frac{l_1 p - \frac{1}{pc_1}}{l_2 p - \frac{1}{pc_2}} = 0$$

$$\text{whence } p = \frac{1}{\sqrt{l_1 c_1}}, \quad \left(l_2 p \neq \frac{1}{pc_2} \right)$$

$$\text{For } \frac{z_1}{z_2} = -4$$

$$\frac{l_1 p - \frac{1}{pc_1}}{l_2 p - \frac{1}{pc_2}} = -4 \text{ whence}$$

$$p = \sqrt{\frac{4c_1 + c_2}{c_1 c_2 (l_1 + 4l_2)}}$$

Thus the filter will pass frequencies between $\frac{1}{2\pi \sqrt{l_1 c_1}}$ and

$$\frac{1}{2\pi \sqrt{\frac{4c_1 + c_2}{c_1 c_2 (l_1 + 4l_2)}}} \text{ without attenuation.}$$

Stewart made several mechanical filters of the above three types. The agreement with the theory is fairly good. Low pass filters were made from two concentric cylinders with space between them and having walls at right angles to the axis of the cylinders. There is a row of apertures in the inner cylinder communicating with the space between the two cylinders. High pass filters were made with a straight tube and short side tubes. Band pass filters are generally the combination of the two types.

The filters may be used with advantage for elimination of needle scratch in a gramophone, microphone and valve noises. They are useful when undesirable frequencies are to be eliminated.

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