U.G. 1st Semester Examination - 2023

MATHEMATICS

[MAJOR]

Course Code : MATH-M-T-1

(Calculus & Analytical Geometry)

[NEP-2020]

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.

1. Answer any **ten** questions : $2\times10=20$

a) If $y = a\cos\log x + b\sin\log x$, then show that $x^2y_2 + xy_1 + y = 0$.

b) Show that the radius of curvature of the parabola $y^2 = 4x$ at the point (0, 0) is 2.

c) Prove that $y = x + \frac{1}{2}$ is an asymptote of the curve $xy^2 - y^2 - x^3 = 0$.

d) Verify whether (0, 1) is a point of inflexion of the curve $x = (\log y)^3$ or not.

e) Show that $y = x^4$ is concave upwards at the origin.

f) Obtain the equation of the straight line $\frac{x}{a} + \frac{y}{b} = 2$, when the origin is shifted to the point (a, b).

[Turn over]

g) For what value of λ, does the equation $xy+5x+\lambda y+15=0$ represent a pair of straight lines.

h) If $I_n=\int_0^{\pi/4} \tan^n \theta d\theta$, then find the value of I_0+I_2.

i) Evaluate $\lim_{x\to\pi/2}(1-\sin x)\tan x$.

j) Find the nature of the conic $r=\frac{1}{4-5\cos\theta}$.

k) Define singular point and characteristic point of a curve $f(x,y,\alpha)=0$ where α is fixed.

l) Find the centre and radius of the sphere $2(x^2+y^2+z^2)-2x+4y-6z=15$.

m) Obtain the equation of a right circular cone whose axis is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$, vertex as the origin and semi-vertical angle is $\pi/4$.

n) Find the length of the arc of the parabola $y^2=4x$ measured from the vertex to one extremity of the latus rectum.

o) Find the value of $\int_0^{\pi/2} \sin^2 x \cos^2 x\, dx$.

2. Answer any four questions : 5×4=20

a) Reduce the equation
$$7x^2-2xy+7y^2-16x+16y-8=0$$
to canonical form and determine the nature of the conic.

b) Find the volume of the solid generated by revolving the cycloid $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$, about real axis.

c) Show that the line $x-1=y-2=z+1$ lies entirely on the surface $x^2-xy+2x+y+2z-1=0$.

d) Prove that the radius of curvature of the catenary $y=a\cosh(x/a)$ at any point is equal in length to the portion of the normal intercepted between the curve and x axis.

e) Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

f) If s be the length of an arc of $3ay^2=x(x-a)^2$ measured from the origin to the point (x,y), then show that $3s^2=4x^2+3y^2$.

3. Answer any two questions : 10×2=20

a) i) Find the equation of the cubic which has the same asymptotes as the curve
$$2x(y-3)^2=3y(x-1)^2$$
and which touches x - axis at the origin and pass through the point (1,1).

ii) If $I_n=\int_0^{\pi/2} \cos^n x\, dx$, then show that
$$I_n=\frac{n-1}{n}I_{n-2},\ n>2.$$
5+5

b) i) Find the equation of the tangent planes to the paraboloid $2x^2 - 3y^2 = 2z$ which pass through the line $\frac{x-1}{3} = \frac{y-1}{-2} = \frac{z}{12}$.

ii) If $y = (x^2 - 1)^n$, then show that $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$.

5+5

c) i) Show that the line $\frac{x+2}{2} = \frac{y}{3} = \frac{z-1}{-2}$ is a generator of the quadric $\frac{x^2}{4} - \frac{y^2}{9} = z$.

ii) Find the envelopes of the family of circles which are described on the double ordinates of the ellipse $x^2/a^2 + y^2/a^2 = 1$ as diameters.

5+5

d) i) Find the area of the portion of the circle $x^2 + y^2 = 1$, which lies inside the parabola $y^2 = 1 - x$.

ii) Show that the locus of the point of intersection of two tangents to $\frac{l}{r} = 1 + e\cos\theta$, which are at right angles to one another is $r^2(e^2-1) - 2ler\cos\theta + 2l^2 = 0$.

5+5